Control Systems Analysis and Design by the Root-Locus Method

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the s plane as the loop gain is varied.

From the design viewpoint, in some systems simple gain adjustment may move the closed-loop poles to desired locations. Then the design problem may become the selection of an appropriate gain value. If the gain adjustment alone does not yield a desired result, addition of a compensator to the system will become necessary.

The closed-loop poles are the roots of the characteristic equation. Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution. (MATLAB provides a simple solution to this problem.) However, just finding the roots of the characteristic equation may be of limited value, because as the gain of the open-loop transfer function varies, the characteristic equation changes and the computations must be repeated.

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the root-locus method, is one in which the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used. Unless otherwise stated, we shall assume that the gain of the open-loop transfer function is the parameter to be varied through all values, from zero to infinity.

By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros. Therefore, it is desired that the designer have a good understanding of the method for generating the root loci of the closed-loop system, both by hand and by use of a computer software program like MATLAB.

In designing a linear control system, we find that the root-locus method proves to be quite useful, since it indicates the manner in which the open-loop poles and zeros should be modified so that the response meets system performance specifications. This method is particularly suited to obtaining approximate results very quickly.

ROOT-LOCUS PLOTS

Angle and Magnitude Conditions. Consider the negative feedback system shown in Figure 9–1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad (1)$$

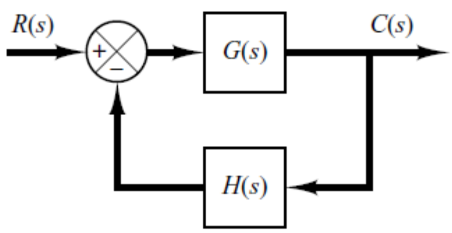

Figure (9 – 1): Control system

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of Equation (6–1) equal to zero. That is,

$$1 + G(s)H(s) = 0$$

Or

$$G(s)H(s) = -1 \qquad (2)$$

Here we assume that $G(s)H(s)$ is a ratio of polynomials in s. [It is noted that we can extend the analysis to the case when $G(s)H(s)$ involves the transport $lage^{-T_s}$] Since $G(s)H(s)$ is a complex quantity, Equ. (2) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\underline{/G(s)H(s)} = \pm 180°(2k + 1) \qquad (k = 0, 1, 2, \dots) \qquad (3)$$

Magnitude condition:

$$|G(s)H(s)| = 1 \qquad (4)$$

The values of s that fulfil both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A locus of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition. The details of applying the angle and magnitude conditions to obtain the closed-loop poles are presented later in this section.

In many cases, $G(s)H(s)$ involves a gain parameter K, and the characteristic equation may be written as

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0$$

Then the root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity.

Note that to begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of $G(s)H(s)$. Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the counter clockwise direction.

For example, if $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)}$$

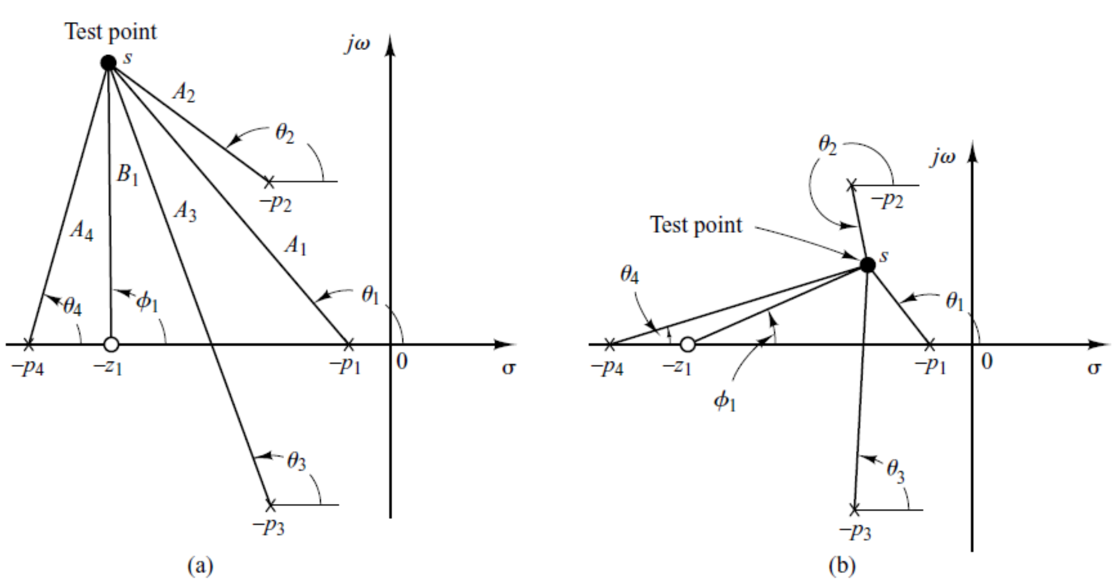

Figure (9–2): (a) and (b) Diagrams showing angle measurements from open-loop poles and open-loop zero to test point s.

where $-p2$ and $-p3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is

$$\underline{/G(s)H(s)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

where $\emptyset_1, \theta_1, \theta_2, \theta_3, and\ \theta_4$ are measured counter clockwise as shown in Figures (9–2)-(a) and (b).The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4}$$

Where $A_1, A_2, A_3, A_4\ and\ B_1$ are the magnitudes of the complex quantities $s+p_1$, $s+p_2$, $s+p_3$, $s+p_4$, and $s+z_1$, respectively, as shown in Figure (9–2)-(a).

Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetrical with respect to this axis. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half s plane.

Illustrative Example 1: Consider the negative feedback system shown in Figure (9–3). (We assume that the value of gain K is nonnegative.) For this system,

$$G(s) = \frac{K}{s(s+1)(s+2)}, \qquad H(s) = 1$$

Let us sketch the root-locus plot and then determine the value of K such that the damping ratio ξ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

For the given system, the angle condition becomes

$$\begin{aligned} \underline{/G(s)} &= \underline{\Big/\frac{K}{s(s+1)(s+2)}} \\ &= -\underline{/s} - \underline{/s+1} - \underline{/s+2} \\ &= \pm 180^\circ(2k+1) \qquad (k = 0, 1, 2, \ldots) \end{aligned}$$

The magnitude condition is

$$|G(s)| = \left|\frac{K}{s(s+1)(s+2)}\right| = 1$$

A typical procedure for sketching the root-locus plot is as follows:

1. *Determine the root loci on the real axis.*

   The first step in constructing a root-locus plot is to locate the open-loop poles, $s = 0, s = -1, and\ s = -2$, in the complex plane. (There are no open-loop zeros in this system.) The locations of the open-loop poles are indicated by crosses. (The locations of the open-loop zeros in this book will be indicated by small circles.) Note that the starting points of the root loci (the points corresponding to K=0) are open-loop poles. The number of individual root loci for this system is three, which is the same as the number of open-loop poles.

To determine the root loci on the real axis, we select a test point, s. If the test point is on the positive real axis, then

$$\underline{/s} = \underline{/s+1} = \underline{/s+2} = 0^\circ$$

This shows that the angle condition cannot be satisfied. Hence, there is no root locus on the positive real axis. Next, select a test point on the negative real axis between $0\ and -1$. Then

$$\underline{/s} = 180°, \qquad \underline{/s+1} = \underline{/s+2} = 0°$$

Thus

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -180°$$

and the angle condition is satisfied. Therefore, the portion of the negative real axis between 0 *and* -1 forms a portion of the root locus. If a test point is selected between -1 *and* -2, then

$$\underline{/s} = \underline{/s+1} = 180°, \qquad \underline{/s+2} = 0°$$

and

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -360°$$

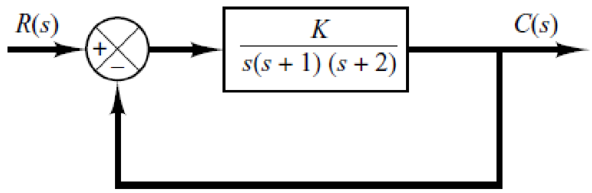

Figure (9–3): Control system

It can be seen that the angle condition is not satisfied. Therefore, the negative real axis from -1 *to* -2 is not a part of the root locus. Similarly, if a test point is located on the negative real axis from -2 *to* $-\infty$, the angle condition is satisfied. Thus, root loci exist on the negative real axis between 0 *and* -1 and between -2 *and* $-\infty$.

2. *Determine the asymptotes of the root loci.*

   The asymptotes of the root loci as s approaches infinity can be determined as follows: If a test point s is selected very far from the origin, then

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3}$$

   and the angle condition becomes

$$-3\underline{/s} = \pm 180°(2k+1) \qquad (k = 0, 1, 2, \dots)$$

   or

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3} \qquad (k = 0, 1, 2, \dots)$$

Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as $60°, -60°, and\ 180°$. Thus, there are three asymptotes. The one having the angle of 180° is the negative real axis.

Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis. Since

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

if a test point is located very far from the origin, then $G(s)$ may be written as

$$G(s) = \frac{K}{s^3 + 3s^2 + \cdots}$$

For large values of s, this last equation may be approximated by

$$G(s) = \frac{K}{(s+1)^3} \qquad (5)$$

A root-locus diagram of $G(s)$ given by Equ. (5) consists of three straight lines. This can be seen as follows: The equation of the root locus is

$$\angle \frac{K}{(s+1)^3} = \pm 180°(2k+1)$$

or

$$-3\angle s+1 = \pm 180°(2k+1)$$

which can be written as

$$\angle s+1 = \pm 60°(2k+1)$$

By substituting $s = \sigma + j\omega$ into this last equation, we obtain

$$\angle \sigma + j\omega + 1 = \pm 60°(2k+1)$$

or

$$\tan^{-1}\frac{\omega}{\sigma+1} = 60°, \qquad -60°, \qquad 0°$$

Taking the *tangent* of both sides of this last equation,

$$\frac{\omega}{\sigma+1} = \sqrt{3}, \qquad -\sqrt{3}, \qquad 0$$

which can be written as

$$\sigma + 1 - \frac{\omega}{\sqrt{3}} = 0, \qquad \sigma + 1 + \frac{\omega}{\sqrt{3}} = 0, \qquad \omega = 0$$

These three equations represent three straight lines, as shown in Figure (9–4). The three straight lines shown are the asymptotes. They meet at point $s = -1$. Thus, the abscissa of the intersection of the asymptotes and the real axis is obtained by setting the denominator of the right-hand side of Equ. (5) equal to zero and solving for s. The asymptotes are almost parts of the root loci in regions very far from the origin.

3. *Determine the breakaway point.*

   To plot root loci accurately, we must find the breakaway point, where the root-locus branches originating from the poles at $0 \; and -1$ break away (as K is increased) from the real axis and move into the complex plane. The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur. [*details in REF: Oqata*]

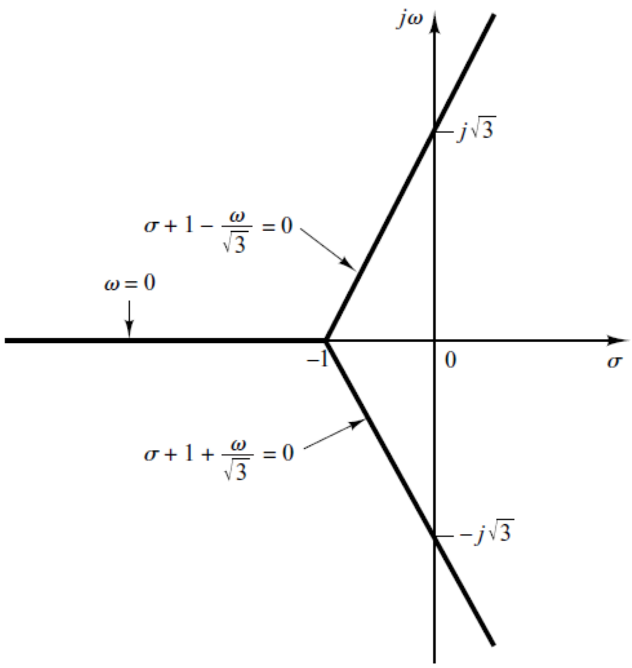

Figure (9–4): Three asymptotes.

For the present example, the characteristic equation $G(s) + 1 = 0$ is given by

$$\frac{K}{s(s+1)(s+2)} + 1 = 0$$

Or

$$K = -(s^3 + 3s^2 + 2s)$$

By setting $dK/ds = 0$, we obtain

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

Or

$$s = -0.4226, \qquad s = -1.5774$$

Since the breakaway point must lie on a root locus between 0 *and* -1, it is clear that $s = -0.4226$ corresponds to the actual breakaway point. Point $s = -1.5774$ is not on the root locus. Hence, this point is not an actual breakaway or break-in point. In fact, evaluation of the values of K corresponding to $s = -0.4226$ *and* $s = -1.5774$ yields

$$K = 0.3849, \qquad \text{for } s = -0.4226$$
$$K = -0.3849, \qquad \text{for } s = -1.5774$$

4. *Determine the points where the root loci cross the imaginary axis.*

   These points can be found by use of Routh's stability criterion as follows: Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

the Routh array becomes

$$\begin{array}{lcc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

The value of K that makes the s^1 term in the first column equal zero is $K = 6$.The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

which yields

$$s = \pm j\sqrt{2}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$

The gain value corresponding to the crossing points is $K = 6$.

An alternative approach is to let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K. For the present system, the characteristic equation, with $s = j\omega$, is

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

or

$$\left(K - 3\omega^2\right) + j\left(2\omega - \omega^3\right) = 0$$

Equating both the real and imaginary parts of this last equation to zero, respectively, we obtain

$$K - 3\omega^2 = 0, \qquad 2\omega - \omega^3 = 0$$

from which

$$\omega = \pm\sqrt{2}, \qquad K = 6 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

Thus, root loci cross the imaginary axis at $\omega = \pm\sqrt{2}$ and the value of K at the crossing points is 6. Also, a root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. The value of K is zero at this point.

5. *Choose a test point in the broad neighbourhood of the $j\omega$ axis and the origin*, as shown in Figure (9–5), and apply the angle condition. If a test point is on the root loci, then the sum of the three angles, $\theta_1 + \theta_2 + \theta_3$, must be 180°. If the test point does not satisfy the angle condition, select another test point until it satisfies the condition. (The sum of the angles at the test point will indicate the direction in which the test point should be moved.) Continue this process and locate a sufficient number of points satisfying the angle condition.

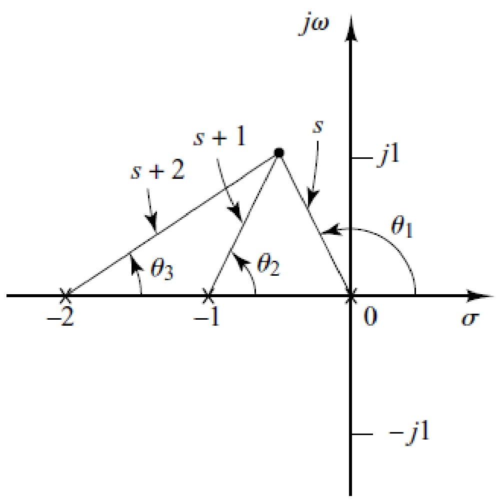

Figure (9–5): Construction of root locus.

6. *Draw the root loci, based on the information obtained in the foregoing steps*, as shown in Figure (9–6).

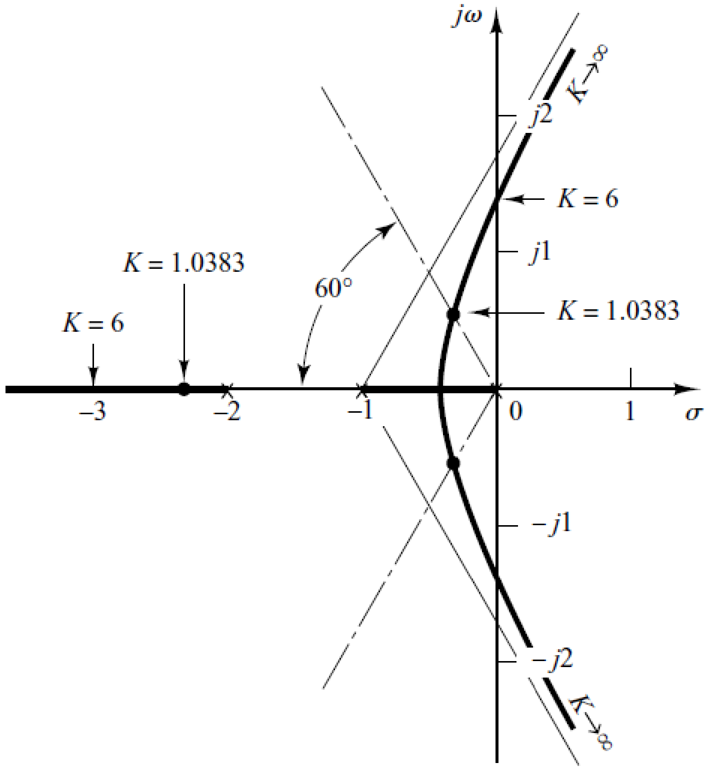

Figure (9–6): Root-locus plot.

7. *Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio* $\xi = 0.5$. Closed-loop poles with $\xi = 0.5$ lie on lines passing through the origin and making the angles $\pm cos^{-1}\xi = \pm cos^{-1}0.5 = \pm 60^0$ with the negative real axis. From Figure (9–6), such closed loop poles having $\xi = 0.5$ are obtained as follows:

$$s_1 = -0.3337 + j0.5780, \qquad s_2 = -0.3337 - j0.5780$$

The value of K that yields such poles is found from the magnitude condition as follows:

$$K = |s(s+1)(s+2)|_{s=-0.3337+j0.5780}$$
$$= 1.0383$$

Using this value of K, the third pole is found at $s = -2.3326$.

Note that, from step 4, it can be seen that for $K = 6$ the dominant closed-loop poles lie on the imaginary axis at $s = \pm j\sqrt{2}$. With this value of K, the system will exhibit sustained oscillations. For $K > 6$, the dominant closed-loop poles lie in the right-half s plane, resulting in an unstable system.

Finally, note that, if necessary, the root loci can be easily graduated in terms of K by use of the magnitude condition. We simply pick out a point on a root locus, measure the magnitudes of the three complex quantities $s, s+1, and\ s+2$, and multiply these magnitudes; the product is equal to the gain value K at that point, or

$$|s| \cdot |s+1| \cdot |s+2| = K$$