TRANSIENT RESPONSE AND STEADY-STATE RESPONSE

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LECTURE 3

4TH STAGE - 1ST SEMESTER - BIOMEDICAL INSTRUMENTATION AND BIOMECHANIC BRANCHES

Lecture 3: Transient Response and Steady-State Response

The time response of a control system consists of two parts:

The transient response and the steady-state response.

By transient response, we mean that which goes from the initial state to the final state.

By steady-state response, we mean the manner in which the system output behaves as t approaches infinity. Thus the system response $c(t)$ may be written as

$$\boldsymbol{c(t) = c_{tr}(t) + c_{ss}(t)}$$

Where:

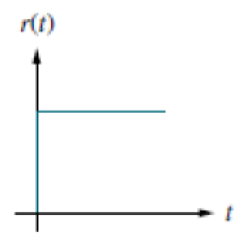
$c_{tr}(t)$: is the transient response
$c_{ss}(t)$: is the steady state response

<u>**Standardized input functions:**</u>

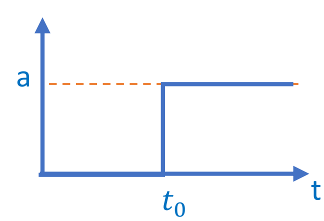
1. **Step function**
   (step position) (unit step function)

$$r(t) = a \quad t \geq 0$$
$$r(t) = 0 \quad t < 0$$

$$r(t) = -a \quad t \geq 0$$
$$r(t) = 0 \quad t < 0$$

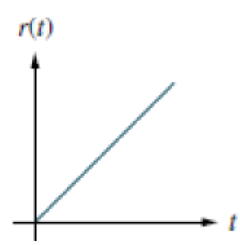


$$r(t) = a \quad t \geq t_0$$
$$r(t) = 0 \quad t < t_0$$



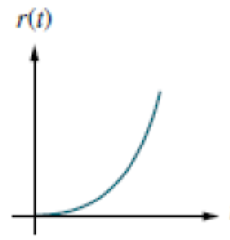
2. **Ramp function**
   (step velocity)

$$r(t) = t \quad t \geq 0$$
$$r(t) = 0 \quad t < 0$$



3. **Parabolic function** (step acceleration)

$$r(t) = t^2 \qquad t \geq 0$$
$$r(t) = 0 \qquad t < 0$$



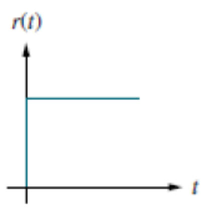
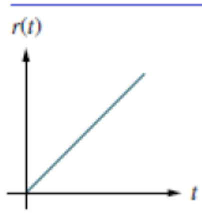
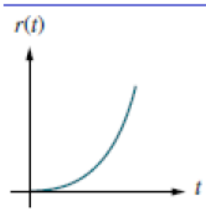
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Figure 15: waveforms for evaluating steady-state errors of position control systems

Transient response:
First-Order Systems

Consider the first-order system shown in Figure 16- (a). Physically, this system may represent an RC circuit, thermal system, or the like.

A simplified block diagram is shown in Figure 16- (b). The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \tag{4}$$

In the following, we shall analyze the system responses to such inputs as the *unit-step, unit-ramp, and unit-impulse* functions. The *initial conditions are assumed to be zero*.

Note that all systems having the same transfer function will exhibit the same output in response to the same input. For any given physical system, the mathematical response can be given a physical interpretation.

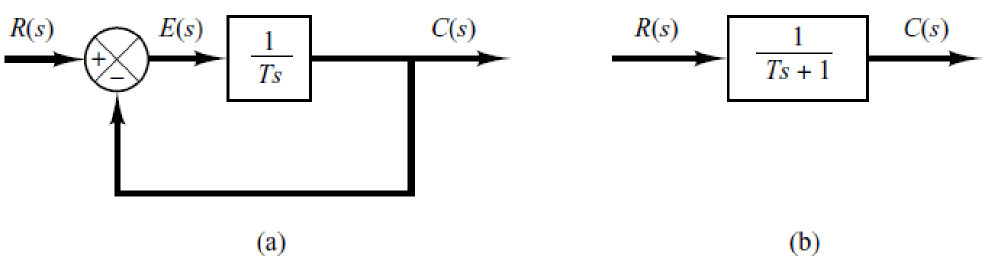


Figure 16: (a) Block diagram of a first-order system; (b) simplified block diagram.

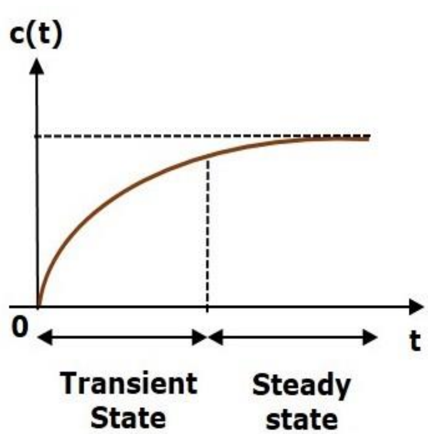


Figure 17: transient and steady-state error

❖ *Unit-Step Response of First-Order Systems*

Since the Laplace transform of the *unit-step function* is $1/s$, substituting $R(s) = 1/s$ into Equation (4), we obtain

$$\boldsymbol{c(s) = \frac{1}{Ts+1}\frac{1}{s}}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{a_1}{s} + \frac{a_2}{Ts+1}$$

$$a_1 = \lim_{s\to 0}\frac{1}{Ts+1} = 1\ ,\ \ a_2 = \lim_{s\to -1/T}\frac{1}{s} = \frac{1}{-1/T} = -T$$

$$\boldsymbol{C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+1/T}} \qquad (5)$$

Taking the inverse Laplace transform of Equation (5), we obtain

$$\boldsymbol{c(t) = 1 - e^{-t/T}\ ,\quad for\ \ t \geq 0} \qquad (6)$$

Equation (6) states that initially the output $\boldsymbol{c(t)}$ is zero and finally it becomes unity.

One important characteristic of such an exponential response curve $\boldsymbol{c(t)}$ is that at $t = T$, the value of $\boldsymbol{c(t)}$ is 0.632, or the response $\boldsymbol{c(t)}$ has reached 63.2% of its total change. This may be easily seen by substituting $t = T$ in $c(t)$.That is,

$$c(t) = 1 - e^{-1} = 0.632$$

Note that the smaller the time constant T, the faster the system response. Another important characteristic of the exponential response curve is that the slope of the tangent line at $t = 0$ is $1/T$, since

$$\left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T}e^{-t/T}\Big|_{t=0} = \frac{1}{T} \qquad (7)$$

The output would reach the final value at $t = T$ if it maintained its initial speed of response. From Equation (7) we see that the slope of the response curve $\boldsymbol{c(t)}$ decreases monotonically from 1/T at $t = 0$ to zero at $t = q$.

The exponential response curve $\boldsymbol{c(t)}$ given by Eq. (6) is shown in Figure 15.

In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At $t = 3T, 4T$, and $5T$, the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for $t \geq 4T$, the response remains within 2% of the final value. As seen from Eq. (6), the steady state is reached mathematically only after an infinite time. In practice, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or four time constants.

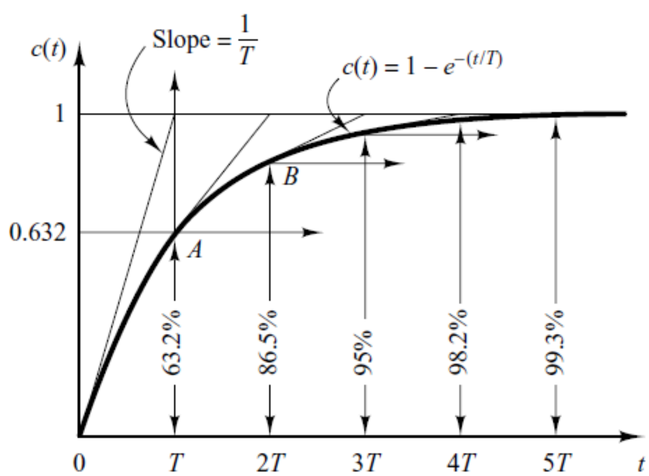


Figure 18: Exponential response curve.

❖ *Unit-Ramp Response of First-Order Systems*

Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of Fig. 16 - (a) as

$$\boldsymbol{c(s) = \frac{1}{Ts+1}\frac{1}{s^2}}$$

Expanding $C(s)$ into partial fractions gives

$$c(s) = \frac{a_1}{s^2} + \frac{a_2}{s} + \frac{a_3}{Ts+1}$$

$$a_1 = \lim_{s\to 0}\frac{1}{Ts+1} = 1\ ,\ a_2 = \lim_{s\to 0}\left(\frac{d}{ds}\left(\frac{1}{Ts+1}\right)\right) = \lim_{s\to 0}\frac{T}{Ts+1} = T,\ a_3 = \lim_{s\to -1/T}\frac{1}{s^2} = \frac{1}{(-1/T)^2} = T^2$$

$$\boldsymbol{c(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}} \qquad (8)$$

Taking the inverse Laplace transform of Eq. (8), we obtain

$$\boldsymbol{c(t) = t - T + Te^{-t/T}}\ , \qquad for\quad t \geq 0 \qquad (9)$$

The error signal $e(t)$ is then

$\boldsymbol{e(t) = r(t) - c(t)}$

$\because \boldsymbol{r(t) = t,\ \ c(t) = t - T + Te^{-t/T}}$

$\therefore \boldsymbol{e(t) = t - \left(t - T + Te^{-t/T}\right)}$

$e(t) = \cancel{t} - \cancel{t} + T - Te^{-t/T}$

$e(t) = T - Te^{-t/T}$

$\therefore \boldsymbol{e(t) = T\left(1 - e^{-t/T}\right)}$

As t approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches T or

$$\boldsymbol{e(\infty) = T}$$

The *unit-ramp input* and the system output are shown in Fig. (16). The error in following the *unit-ramp input* is equal to T for sufficiently large t. The smaller the time constant T, the smaller the steady-state error in following the ramp input.

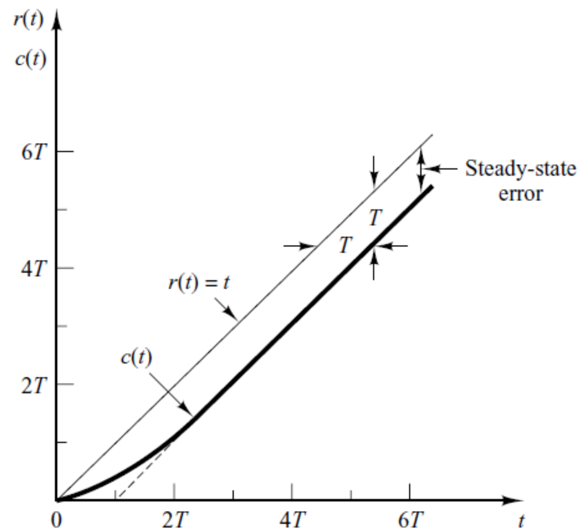


Figure 19: Unit-ramp response of the system shown in Fig. 16(a).

❖ *Unit-Impulse Response of First-Order Systems*

For the unit-impulse input, $R(s) = 1$ and the output of the system of Fig. 16 - (a) can be obtained as

$$c(s) = \frac{1}{Ts+1} \tag{10}$$

The inverse Laplace transform of Eq. 10, gives

$$c(t) = \frac{1}{T}e^{-t/T}, \quad for\ t \geq 0 \tag{11}$$

The response curve given by Eq. (11) is shown in Fig. 20

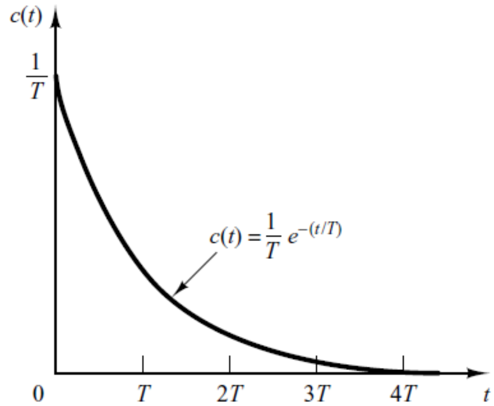


Figure 20: Unit-impulse response of the system shown in Fig. 16 - (a).

An Important Property of Linear Time-Invariant Systems

In the analysis above, it has been shown that for the unit-ramp input the output $c(t)$ is

$$\boldsymbol{c(t) = t - T + Te^{-t/T}}\ , \quad for\ t \geq 0$$

For the unit-step input, which is the derivative of unit-ramp input, the output $c(t)$ is

$$\boldsymbol{c(t) = 1 - e^{-t/T}\ , \quad for\ t \geq 0}$$

Finally, for the unit-impulse input, which is the derivative of unit-step input, the output

$$\boldsymbol{c(t) = \frac{1}{T}e^{-t/T}, \quad for\ t \geq 0}$$

Comparing the system responses to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition. This is a property of linear time-invariant systems.