TRANSIENT RESPONSE: SECOND- ORDER SYSTEMS

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LECTURE 4

4TH STAGE - 1ST SEMESTER - BIOMEDICAL INSTRUMENTATION AND BIOMECHANIC BRANCHES

Transient response:
Second- Order Systems

We consider a servo system as an example of a second-order system.

Servo System:

The servo system shown in Fig.21- (a) consists of a proportional controller and load elements (inertia and viscous-friction elements). Suppose that we wish to control the output position c in accordance with the input position r.

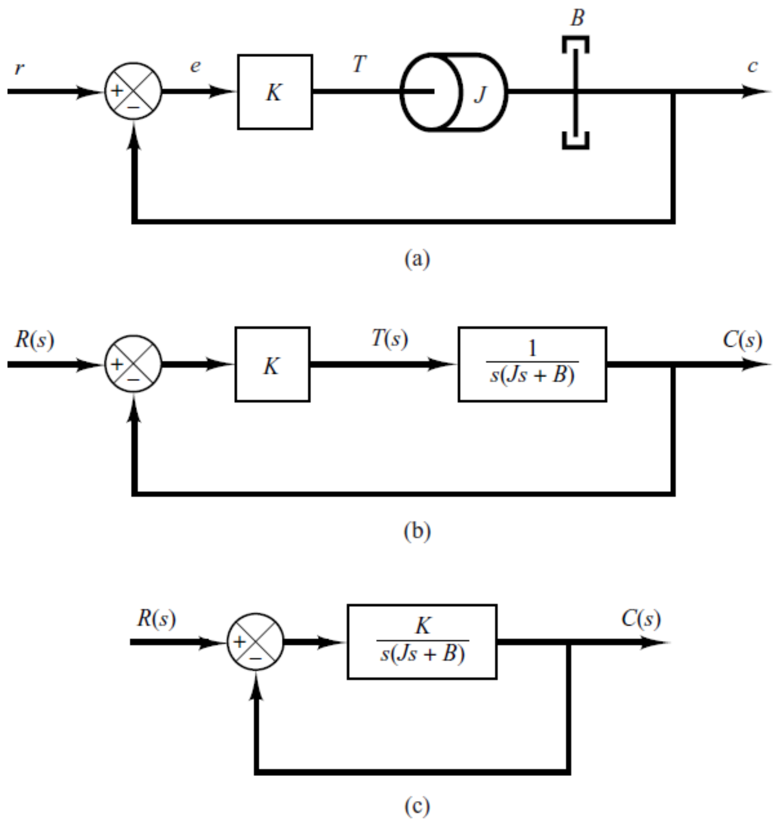

Figure (21): (a) Servo system; (b) block diagram; (c) simplified block diagram.

The equation for the load elements is

$$J\ddot{c} + B\dot{c} = T$$

where T is the torque produced by the proportional controller whose gain is K. By taking Laplace transforms of both sides of this last equation, assuming the zero initial conditions, we obtain

$$Js^2C(s) + BsC(s) = T(s)$$

So the transfer function between $C(s)$ and $T(s)$ is

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js+B)}$$

By using this transfer function, Fig.21- (a) can be redrawn as in Fig.21 - (b), which can be modified to that shown in Fig.21 - (c). The closed-loop transfer function is then obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+Bs+K} = \frac{K/J}{s^2+(B/J)s+(K/J)}$$

Such a system where the closed-loop transfer function possesses two poles is called a second-order system. (*Some second-order systems may involve one or two zeros*).

❖ *Step Response of Second-Order System.*

The closed-loop transfer function of the system shown in Fig.18 - (c) is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+Bs+K} \qquad (4\text{-}1)$$

which can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2+(B/J)s+(K/J)} = \frac{K/J}{\left[s+\frac{B}{2J}+\sqrt{\left(\frac{B}{2J}\right)^2-\frac{K}{J}}\right]\left[s+\frac{B}{2J}-\sqrt{\left(\frac{B}{2J}\right)^2-\frac{K}{J}}\right]}$$

The closed-loop poles are complex conjugates if $B^2-4KJ<0$ and they are real

if $B^2-4KJ\geq 0$. In the transient-response analysis, it is convenient to write

$$\boldsymbol{\frac{K}{J} = \omega_n^2\,, \frac{B}{J} = 2\xi\omega_n = 2\sigma}$$

where $\boldsymbol{\sigma}$ is called the *attenuation*; $\boldsymbol{\omega_n}$, the *undamped natural frequency*; and **ξ**, the *damping ratio* of the system. The damping ratio ξ is the ratio of the actual damping B to the critical damping $B_c = 2\sqrt{JK}$ or

$$\xi = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

In terms of ξ and $\boldsymbol{\omega_n}$, the system shown in Fig.18 - (c) can be modified to that shown in Fig.22, and the closed-loop transfer function $C(s)/R(s)$ given by Eq. (4-1) can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \tag{4-2}$$

This form is called the standard form of the second-order system.

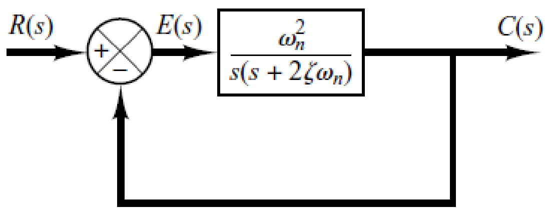

Figure 22: Second-order system

The dynamic behavior of the second-order system can then be described in terms of two parameters ξ and ω_n.

- If $\mathbf{0} < \boldsymbol{\xi} < \mathbf{1}$, the closed-loop poles are ***complex conjugates*** and lie in the left-half s-plane. The system is then called ***underdamped***, and the transient response is *oscillatory*.
- If $\boldsymbol{\xi} = \mathbf{0}$, the transient response does not *die out*.
- If $\boldsymbol{\xi} = \mathbf{1}$, the system is called ***critically damped***.
- ***Overdamped*** systems correspond to $\boldsymbol{\xi} > \mathbf{1}$.

For Unit step input, we consider three different cases:

The underdamped $(0 < \xi < 1)$, critically damped $(\xi = 1)$, and overdamped $(\xi > 1)$ cases

1) ***Underdamped case*** (0<ξ<1):

In this case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \xi\omega_n + j\omega_d)(s + \xi\omega_n - j\omega_d)}$$

Where $\omega_d = \omega_n\sqrt{1 - \xi^2}$. The frequency ω_d is called the damped natural frequency. For a unit-step input, $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\xi\omega_n s + \omega_n^2)s} \qquad (4\text{-}3)$$

The inverse Laplace transform of Eq. (4-3) can be obtained easily if $C(s)$ is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Referring to the Laplace transform table (4 -1), it can be shown that

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$
$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

Hence the inverse Laplace transform of Eq.(4 -3) is obtained as

$$\mathcal{L}^{-1}[C(s)] = c(t)$$

$$= 1 - e^{-\xi\omega_n t}\left(cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \quad , for\ t \geq 0 \qquad (4-4)$$

From Eq. (4 - 4), it can be seen that the frequency of transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ξ. The error signal for this system is the difference between the input and output and is

$$e(t) = r(t) - c(t)$$
$$= e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right), \quad \text{for } t \geq 0$$

This error signal exhibits a damped sinusoidal oscillation. At steady state, or at $t = \infty$, no error exists between the input and output.

2) ***Critically damped case*** ($\xi = 1$):

If the two poles of $C(s)/R(s)$ are equal, the system is said to be a critically damped one.

For a unit-step input, R(s)=1/s and C(s) can be written

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s} \qquad (4-5)$$

The inverse Laplace transform of Eq.(4 – 5) may be found as

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \ \ , for\ t \geq 0 \qquad (4-6)$$

This result can also be obtained by letting z approach unity in Eq.(4 – 4) and by using the following limit:

$$\lim_{\zeta\to 1}\frac{\sin\omega_d t}{\sqrt{1-\zeta^2}} = \lim_{\zeta\to 1}\frac{\sin\omega_n\sqrt{1-\zeta^2}\,t}{\sqrt{1-\zeta^2}} = \omega_n t$$

3) *Overdamped case* ($\xi > 1$):

In this case, the two poles of C(s)/R(s) are negative real and unequal.

For a unit-step input, R(s)=1/s and C(s) can be written

$$C(s) = \frac{\omega_n^2}{\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)s} \qquad (4-7)$$

The inverse Laplace transform of Eq.(4 – 6) is

$$\begin{aligned} c(t) &= 1 + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)} e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)} e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0 \end{aligned} \qquad (4-8)$$

Where $s_1 = \left(\xi + \sqrt{\xi^2-1}\right)\omega_n$ *and* $s_2 = \left(\xi - \sqrt{\xi^2-1}\right)\omega_n$

Thus, the response $c(t)$ includes two decaying exponential terms.

When ξ is appreciably greater than unity, one of the two decaying exponentials decreases much faster than the other, so the faster-decaying exponential term (which corresponds to a smaller time constant) may be neglected.

A family of unit-step response curves $c(t)$ with various values of ξ is shown in Fig.(23), where the abscissa is the dimensionless variable $w_n t$. The curves are functions only of ξ. These curves are obtained from Eq. (4 – 4), (4 – 6), and (4 – 8). The system described by these equations was initially at rest.

Note that two second-order systems having the same ξ but different $\boldsymbol{\omega_n}$ will exhibit the same *overshoot* and the same *oscillatory pattern*. Such systems are said to have the same *relative stability*.

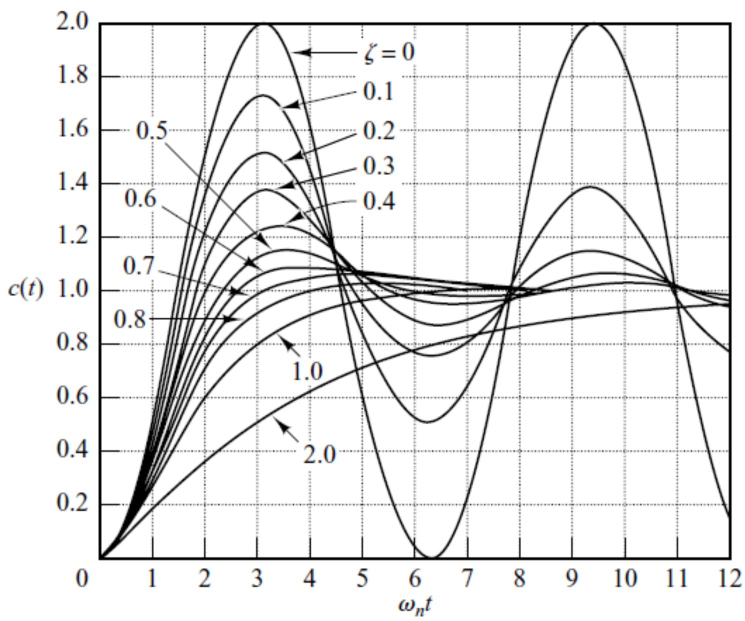

Figure (23): Unit-step response curves of the system shown in Fig. (19)

Table (4 – 1): Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1}e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

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Table (4 – 1): Laplace Transform Pairs (continued)

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n\sqrt{1-\zeta^2}\,t \quad (0<\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1, \quad 0<\phi<\pi/2)$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1, \quad 0<\phi<\pi/2)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
25	$1 - \cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega} t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Table (4 – 2): Properties of Laplace Transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_{\pm}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_{\pm}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_{\pm}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathcal{L}_{\pm}\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$ if $\int_0^\infty f(t)\,dt$ exists
10	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
11	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \quad \alpha \geq 0$
12	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathcal{L}[t^nf(t)] = (-1)^n \frac{d^n}{ds^n}F(s) \quad (n = 1, 2, 3, \dots)$
15	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$ if $\lim_{t\to 0} \frac{1}{t}f(t)$ exists
16	$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$