TRANSIENT RESPONSE SPECIFICATIONS

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LECTURE 5

4TH STAGE - 1ST SEMESTER - BIOMEDICAL INSTRUMENTATION AND BIOMECHANIC BRANCHES

Transient-Response Specifications

Definitions of Transient-Response Specifications

Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input, since it is easy to generate and is sufficiently drastic. (If the response to a step input is known, it is mathematically possible to compute the response to any input.)

The transient response of a system to a unit-step input depends on the initial conditions. For convenience in comparing transient responses of various systems, it is a common practice to use the standard initial condition that the system is at rest initially with the output and all time derivatives thereof zero. Then the response characteristics of many systems can be easily compared.

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

- **Delay time, t_d.**
- **Rise time, t_r.**
- **Peak time, t_p.**
- **Maximum overshoot, M_p.**
- **Settling time, t_s.**

These specifications are defined in what follows and are shown graphically in Fig. (21).

1. **Delay time**, t_d:The delay time is the time required for the response to reach half the final value the very first time.
2. **Rise time**, t_r:The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second-order systems, the 0% to 100%rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

3. **Peak time**, t_p:The peak time is the time required for the response to reach the first peak of the overshoot.
4. **Maximum (percent) overshoot**, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

5. **Settling time**, t_s: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system. Which percentage error criterion to use may be determined from the objectives of the system design in question.

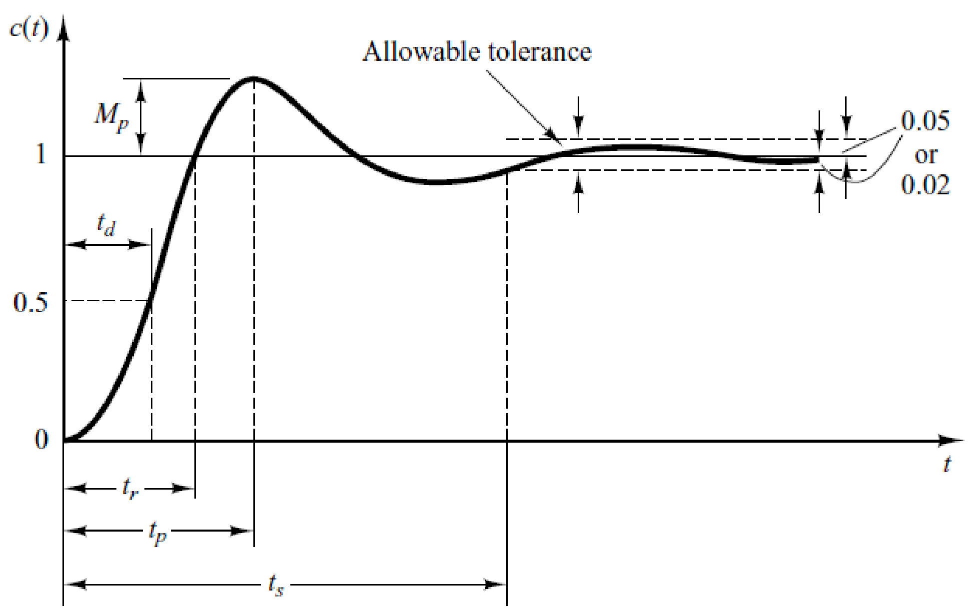

Figure (21): Unit-step response curve showing t_d, t_r, t_p, M_p, and t_s.
[Ref., '*Ogata, K., 2010. Modern control engineering (Vol. 5). Upper Saddle River, NJ: Prentice hall.*]

A Few Comments on Transient-Response Specifications.

Note that not all these specifications necessarily apply to any given case. For example, for an overdamped system, the terms peak time and maximum overshoot do not apply. (For systems that yield steady-state errors for step inputs, this error must be kept within a specified percentage level.

Except for certain applications where oscillations cannot be tolerated, it is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a desirable transient response of a second-order system, the damping ratio must be between 0.4 and 0.8. Small values of ξ (that is, $\xi < 0.4$) yield excessive overshoot in the transient response, and a system with a large value of ξ (that is, $\xi > 0.8$) responds sluggishly.

We shall see later that the maximum overshoot and the rise time conflict with each other. In other words, both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one of them is made smaller, the other necessarily becomes larger.

Second-Order Systems and Transient-Response Specifications

In the following, we shall obtain *the rise time, peak time, maximum overshoot, and settling time* of the second-order system given by Equation (5-1). These values will be obtained in terms of ξ and ω_n. The system is assumed to be underdamped.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \qquad (5-1)$$

❖ ***Rise time*** $\boldsymbol{t_r}$:

Referring to Equation (4 – 4), we obtain the rise time $\boldsymbol{t_r}$ by letting $c(\boldsymbol{t_r}) = \mathbf{1}$

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right) \qquad (5-2)$$

Since $e^{-\xi\omega_n t_r} \neq 0$, we obtain the following equation:

$$cos\omega_d t_r + \frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d t_r = 0 \qquad (5-3)$$

Since $\omega_n\sqrt{1-\xi^2} = \omega_d$ and $\xi\omega_n = \sigma$, we have

$$tan\omega_d t_r = -\frac{\sqrt{1-\xi^2}}{\xi} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time $\boldsymbol{t_r}$ is

$$\boldsymbol{t_r = \frac{1}{\omega_d} tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}} \qquad (5-4)$$

where angle β is defined in Figure (22). Clearly, for a small value of t_r and ω_d must be large.

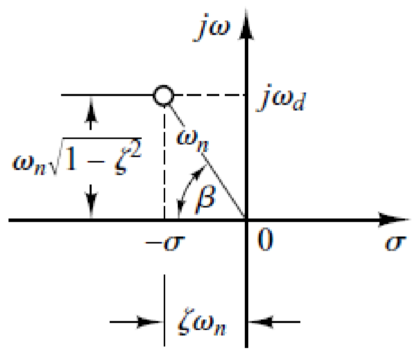

Figure (22): Definition of the angle β.

❖ ***Peak time*** $\boldsymbol{t_p}$:

Referring to Eq. (4 – 4), we may obtain the peak time by differentiating $c(t)$ with respect to time and letting this derivative equal zero. Since

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

and the cosine terms in this last equation cancel each other, dc/dt evaluated at $t = t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin\omega_d t_p = 0$$

Or $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$

Since the peak time corresponds to the first peak overshoot $\omega_d t_p = \pi$, Hence

$$\boldsymbol{t_p = \frac{\pi}{\omega_d}} \qquad (5-5)$$

The peak time $\boldsymbol{t_p}$ corresponds to one-half cycle of the frequency of damped oscillation.

❖ ***Maximum overshoot M_p:***

The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, M_p is obtained as

$$\begin{aligned} M_p &= c(t_p) - 1 \\ &= -e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) \\ &= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \end{aligned} \qquad (5-6)$$

The maximum percent overshoot is $\boldsymbol{e^{-(\sigma/\omega_d)\pi} \times 100\%}$.

If the final value $c(\infty)$ of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

❖ ***Settling time t_s*** :

For an underdamped second-order system, the transient response is obtained as

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0$$

The curves $1 \pm \left(e^{-\xi\omega_n t}/\sqrt{1-\xi^2}\right)$ are the envelope curves of the transient response to a unit-step input. The response curve c(t) always remains within a pair of the envelope curves, as shown in Figure (23). The time constant of these envelope curves is $1/\xi\omega_n$.

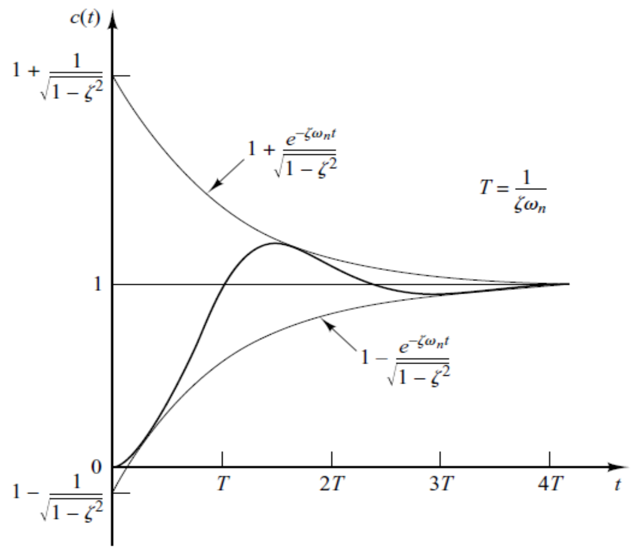

Figure (23): Pair of envelope curves for the unit step response curve of the system shown in Fig. (19).

The speed of decay of the transient response depends on the value of the time constant $1/\xi\omega_n$. For a given ω_n, the settling time t_s is a function of the damping ratio ξ.

The settling time corresponding to $a \pm 2\% \, or \, ; 5\%$ tolerance band may be measured in terms of the time constant $T = 1/\xi\omega_n$ from the curves of Figure (20) –LEC4 for different values of ξ. The results are shown in Figure (24). For $0 < \xi < 0.9$, if the 2% criterion is used, t_s is approximately four times the time constant of the system. If the 5% criterion is used, then t_s is approximately three times the time constant. Note that the settling time reaches a minimum value around $\xi = 0.76$ (for the 2% criterion) or $\xi = 0.68$ (for the 5% criterion) and then increases almost linearly for large values of ξ. The discontinuities in the curves of Figure (24) arise because an infinitesimal change in the value of ξ can cause a finite change in the settling time.

For convenience in comparing the responses of systems, we commonly define the settling time ts to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\xi\omega_n} \qquad (2\%\ creterion)$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\xi\omega_n} \qquad (5\%\ creterion)$$

Note that the settling time is inversely proportional to the product of the damping ratio and the undamped natural frequency of the system. Since the value of ξ is usually determined from the requirement of permissible maximum overshoot, the settling time is determined primarily by the undamped natural frequency ω_n. This means that the duration of the transient period may be varied, without changing the maximum overshoot, by adjusting the undamped natural frequency ω_n.

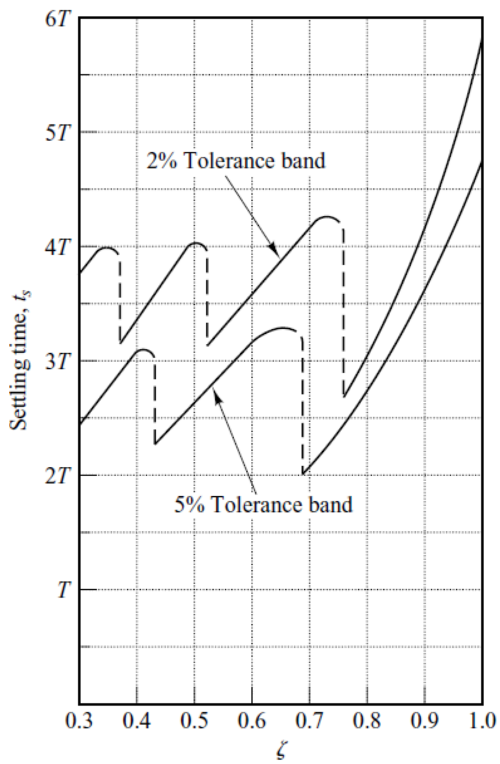

Figure (24): Settling time t_s versus ξ curves.

From the preceding analysis, it is evident that for rapid response ω_n must be large. To limit the maximum overshoot M_p and to make the settling time small, the damping ratio ξ should not be too small. The relationship between the maximum percent overshoot M_p and the damping ratio ξ is presented in Figure (25). Note that if the damping ratio is between 0.4 and 0.7, then the maximum percent overshoot for step response is between 25% and 4%.

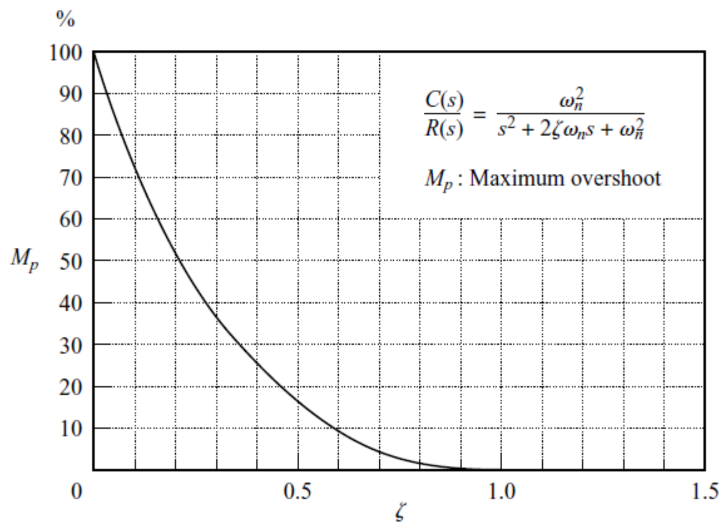

Figure (25): M_p versus ξ curve.

It is important to note that the equations for obtaining the rise time, peak time, maximum overshoot, and settling time are valid only for the standard second-order system defined by Equation (5–1). If the second-order system involves a zero or two zeros, the shape of the unit-step response curve will be quite different from those shown in Figure (20) – LEC4.

Table (5 – 1): Damping ratio of second order system