Examples: Transient-Response Specifications

Second-Order Systems and Transient-Response Specifications

The system is assumed to be underdamped.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

- ***Rise time*** t_r :

$$t_r = \frac{1}{\omega_d} tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}$$

- ***Peak time*** t_p**:**

$$t_p = \frac{\pi}{\omega_d}$$

The peak time t_p corresponds to one-half cycle of the frequency of damped oscillation.

- ***Maximum overshoot*** M_p**:**

The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, M_p is obtained as

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

- ***Settling time*** t_s :

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\xi\omega_n} \qquad (2\%\ creterion)$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\xi\omega_n} \qquad (5\%\ creterion)$$

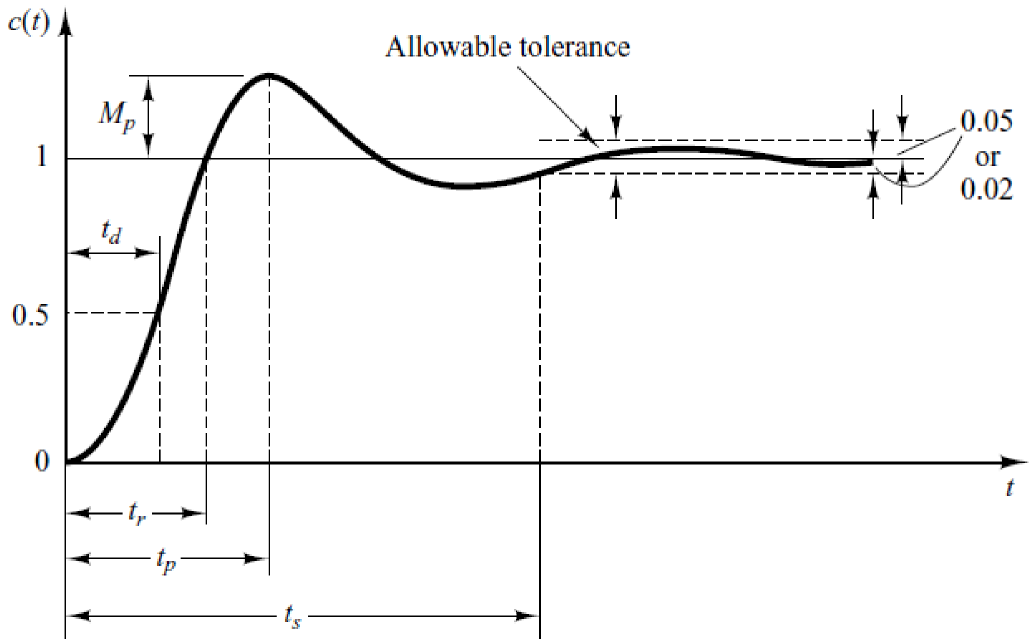

Figure (21): Unit-step response curve showing $\boldsymbol{t_d}$, $\boldsymbol{t_r}$, $\boldsymbol{t_p}$, $\boldsymbol{M_p}$, and $\boldsymbol{t_s}$.
[Ref., '*Ogata, K., 2010. Modern control engineering (Vol. 5). Upper Saddle River, NJ: Prentice hall.*]

Example 1:

Consider the second order system in equation below, where $\xi = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time t_r, peak time t_p, maximum overshoot Mp and settling time t_s when the system is subjected to a *unit-step input*.

Solution: From the given values of ξ and ω_n, we obtain:

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 4\ ,\ \ \sigma = \xi\omega_n = 3$$

Rise time t_r:

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time t_r is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p:

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p:

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling time t_s:

For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

Example 2:

The system shown in Figure (6 – 1) - (a) is subjected to a unit-step input, the system output responds as shown in Figure (6 – 2) - (b). Determine the values of K and T from the response curve.

Solution. The maximum overshoot of $M_p = 25.4\%$ corresponds to $\xi = 0.4$.

From the response curve we have $t_p = 3$

Consequently,

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3$$

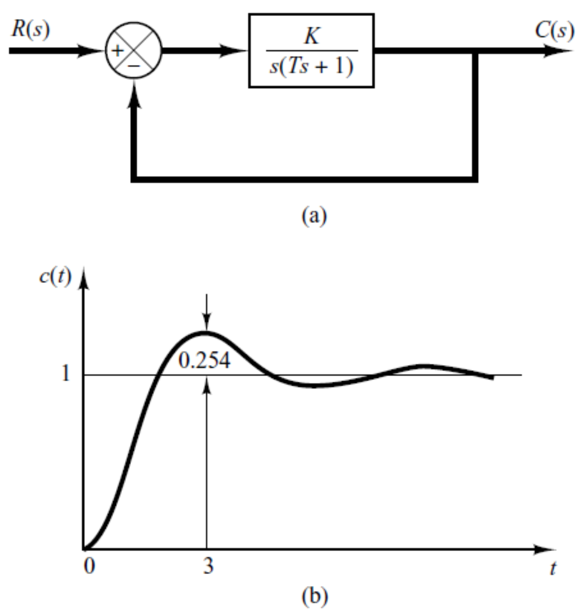

Figure (6-1): (a) Closed-loop system; (b) unit-step response curve.

It follows that $\omega_n = 1.4$

From the block diagram we have

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

from which

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$

Therefore, the values of T and K are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$

Example 3:

Determine the values of K and k of the closed-loop system shown in Figure (6 – 2) so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J=1 kg-m2.

Solution: The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K}$$

By substituting $J = 1\ kg - m2$ into this last equation, we have

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K}$$

Note that in this problem

$$\omega_n = \sqrt{K}, \qquad 2\zeta\omega_n = Kk$$

The maximum overshoot M_p is

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

which is specified as 25%. Hence

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.25$$

from which

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.386$$

Or, $\xi = 0.404$

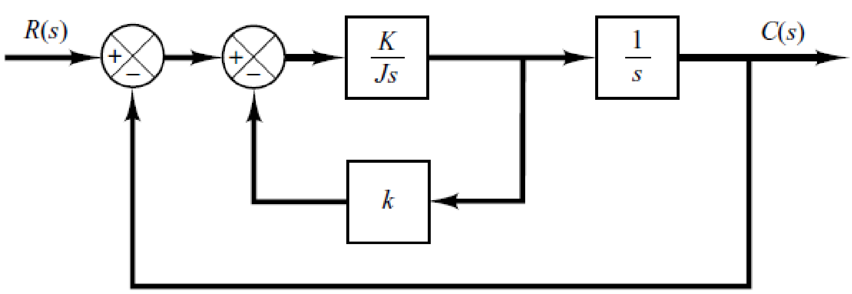

Figure (6 – 2): Closed-loop system.

The peak time t_p is specified as $2\ sec$. And so

$$t_p = \frac{\pi}{\omega_d} = 2$$

Or

$$\omega_d = 1.57$$

Then the undamped natural frequency ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{1.57}{\sqrt{1 - 0.404^2}} = 1.72$$

Therefore, we obtain

$$K = \omega_n^2 = 1.72^2 = 2.95 \text{ N-m}$$

$$k = \frac{2\zeta\omega_n}{K} = \frac{2 \times 0.404 \times 1.72}{2.95} = 0.471 \text{ sec}$$

<u>Homework:</u>

Write the transfer function of the system in Examples (1-2-3) then Plot the transient response of the second order system based on its specifications.