Steady-State response in unity feedback control systems

The steady state error is a measure of system accuracy. Errors in control system can be attributed to many factors such as the nature of the inputs, system type and from nonlinearities of system components. For example; Changes in the reference input will cause unavoidable errors during transient periods and may also cause steady-state errors. Imperfections in the system components, such as static friction, backlash, and amplifier drift, as well as aging or deterioration, will cause errors at steady state. We shall investigate a type of steady-state error that is caused by the incapability of a system to follow particular types of inputs.

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. (The only way we may be able to eliminate this error is to modify the system structure.) Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system.

Steady-State Errors

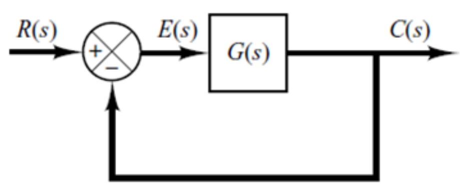


Figure (7–7): Control system.

Consider the system shown in Figure (7 – 7). The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \Rightarrow C(s) = \frac{G(s)R(s)}{1 + G(s)}$$

$$C(s) = E(s)G(s) \Rightarrow \frac{\cancel{G}(s)R(s)}{1+G(s)} = E(s)\cancel{G}(s) \Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

where the error $e(t)$ is the difference between the input signal and the output signal.

The final-value theorem provides a convenient way to find the steady-state performance of a stable system. Since E(s) is

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

the steady-state error is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

The static error constants defined in the following are figures of merit of control systems. The higher the constants, the smaller the steady-state error. In a given system, the output may be the position, velocity, pressure, temperature, or the like. The physical form of the output, however, is immaterial to the present analysis. Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on. This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on.

Steady state error of standard inputs

❖ Static Position Error Constant K_p

The steady-state error of the system for a unit-step input $r(t) = 1 \Rightarrow R(s) = 1/s$ is

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s\to 0} \frac{s.\frac{1}{s}}{1 + G(s)} = \lim_{s\to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \lim\limits_{s\to 0} G(s)} = \frac{1}{1 + G(0)}$$

The static position error constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

Thus, the steady-state error in terms of the static position error constant K_p is given by

$$\therefore \boldsymbol{e_{ss}} = \frac{\mathbf{1}}{\mathbf{1} + \boldsymbol{K_p}}$$

For a type 0 system,

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For example; $G(s) = \frac{1}{s+1}$, $G(s) = \frac{1}{s^2+s+1}$

For a $type$ 1 or higher system,

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

For example; ; $G(s) = \frac{1}{s(s+1)} \Rightarrow e_{ss} = \frac{1}{1+\infty} = 0$

Hence, for a $type$ 0 system, the static position error constant K_p is finite, while for a $type$ 1 or higher system, K_p is infinite.

For a unit-step input, the steady-state error e_{ss} may be summarized as follows:

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

From the foregoing analysis, it is seen that the response of a feedback control system to a step input involves a steady-state error if there is no integration in the feedforward path. (If small errors for step inputs can be tolerated, then a $type$ 0 system may be permissible, provided that the gain K is sufficiently large. If the gain K is too large, however, it is difficult to obtain reasonable relative stability.) If zero steady-state error for a step input is desired; the type of the system must be one or higher.

❖ Static Velocity Error Constant K_v

The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s\to 0} \frac{s.\frac{1}{s^2}}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s\big(1 + G(s)\big)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

The static velocity error constant K_v is defined by

$$K_v = \lim_{s\to 0} sG(s)$$

Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$\therefore\, \boldsymbol{e_{ss} = \frac{1}{K_v}}$$

The term *velocity error* is used here to express the steady-state error for a ramp input. The dimension of the velocity error is the same as the system error. That is, velocity error is not an error in velocity, but it is an error in position due to a ramp input.

For a $type$ 0 system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a $type$ 1 system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For a $type$ 2 or higher system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

The steady-state error e_{ss} for the unit-ramp input can be summarized as follows:

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

The foregoing analysis indicates that a type 0 system is incapable of following a ramp input in the steady state. The type 1 system with unity feedback can follow the ramp input with a finite error. In steady-state operation, the output velocity is exactly the same as the input velocity, but there is a positional error. This error is proportional to the velocity of the input and is inversely proportional to the gain K.

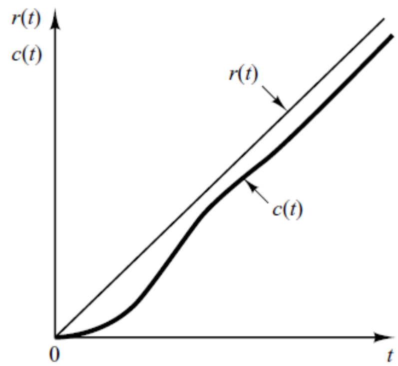


Figure (7–8): Response of a type 1 unity-feedback system to a ramp input.

Figure (7–8) shows an example of the response of a $type$ 1 system with unity feedback to a ramp input. The $type$ 2 or higher system can follow a ramp input with zero error at steady state.

❖ Static Acceleration Error Constant K_a

The steady-state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$r(t) = \frac{t^2}{2}, \quad \text{for } t \geq 0$$
$$= 0, \quad \text{for } t < 0$$

is given by

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s.\frac{1}{s^3}}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2\big(1 + G(s)\big)} = \frac{1}{\lim\limits_{s \to 0} s^2 G(s)}$$

The static acceleration error constant K_a is defined by the equation

$$K_a = \lim_{s \to 0} s^2 G(s)$$

The steady-state error is then

$$\therefore \boldsymbol{e_{ss} = \frac{1}{K_a}}$$

Note that the acceleration error, the steady-state error due to a parabolic input, is an error in position.

The values of $\boldsymbol{K_a}$ are obtained as follows:

For a type 0 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 1 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 2 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For a type 3 or higher system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Thus, the steady-state error for the unit parabolic input is

$$e_{ss} = \infty, \qquad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \qquad \text{for type 2 systems}$$

$$e_{ss} = 0, \qquad \text{for type 3 or higher systems}$$

Note that both *type 0* and *type 1* systems are incapable of following a parabolic input in the steady state. The *type 2* system with unity feedback can follow a parabolic input with a finite error signal. Figure (7–8) shows an example of the response of a *type 2* system with unity feedback to a parabolic input. The *type 3* or higher system with unity feedback follows a parabolic input with zero error at steady state.

Table (7–1) summarizes the steady-state errors for type 0, type 1, and type 2 systems when they are subjected to various inputs. The finite values for steady-state errors appear on the diagonal line. Above the diagonal, the steady-state errors are infinity; below the diagonal, they are zero.

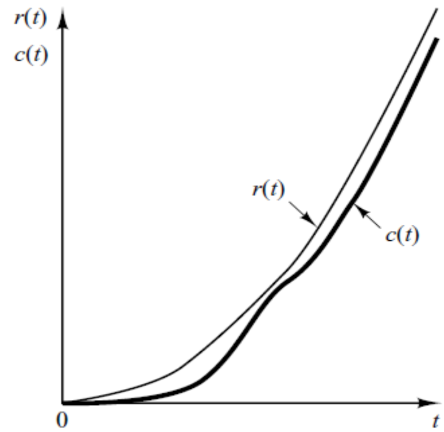


Figure (7–8): Response of a type 2 unity-feedback system to a parabolic input.

Table (7 – 1): Steady-State Error in Terms of Gain K

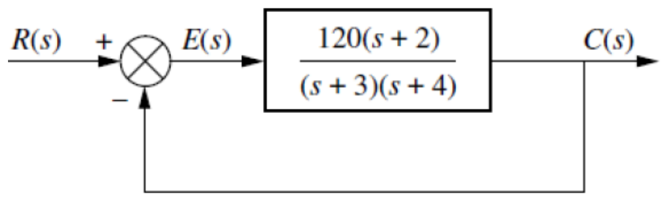
	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Remember that the terms position error, velocity error, and acceleration error mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out, the input and output move at the same velocity but have a finite position difference.

The error constants $\boldsymbol{K_p, K_v, and\ K_a}$ describe the ability of a unity-feedback system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance. It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range. It is noted that to improve the steady-state performance we can increase the type of the system by adding an integrator or integrators to the feedforward path. This, however, introduces an additional stability problem. The design of a satisfactory system with more than two integrators in series in the feedforward path is generally not easy.

Example: Find the steady-state errors for inputs of $5u(t), 5tu(t)$, and $5t^2u(t)$ to the system shown below. The function $u(t)$ is the *unit step*.

A.



For step input $5u(t)$, we must calculate the position error coefficient ($\boldsymbol{K_p}$):

$$K_p = \lim_{s\to 0} G(s) = \frac{120 \times (0+2)}{(0+3)\times(0+4)} = \frac{240}{12} = 20$$

$$\boldsymbol{e_{ss}} = \frac{\mathbf{5}}{\mathbf{1}+\boldsymbol{K_p}} = \frac{\mathbf{5}}{\mathbf{1+20}} = \frac{\mathbf{5}}{\mathbf{21}}$$

For ramp input $5tu(t)$, we must calculate the velocity error coefficient ($\boldsymbol{K_v}$):

$$K_v = \lim_{s\to 0} sG(s) = \frac{(0)\times 120 \times (0+2)}{(0+3)\times(0+4)} = \frac{0}{12} = 0$$

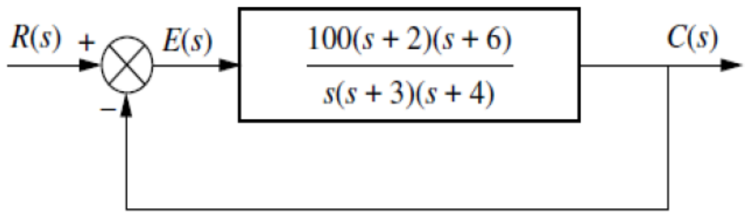
$$\boldsymbol{e_{ss}} = \frac{\mathbf{5}}{\boldsymbol{K_v}} = \frac{\mathbf{5}}{\mathbf{0}} = \infty$$

For parabolic input $5t^2u(t)$, we must calculate the acceleration error coefficient ($\boldsymbol{K_a}$):

$$K_a = \lim_{s\to 0} s^2G(s) = \frac{(0)\times 120 \times (2)}{(3)\times(4)} = \frac{0}{12} = 0$$

$$\boldsymbol{e_{ss}} = \frac{\mathbf{5\times 2}}{\boldsymbol{K_a}} = \frac{\mathbf{10}}{\mathbf{0}} = \infty$$

B.



For step input $5u(t)$, we must calculate the position error coefficient ($\boldsymbol{K_p}$)**:**

$$K_p = \lim_{s \to 0} G(s) = \frac{100 \times (0+2) \times (0+6)}{0 \times (0+3) \times (0+4)} = \infty$$

$$\boldsymbol{e_{ss} = \frac{5}{1+K_p} = \frac{5}{1+\infty} = 0}$$

For ramp input $5tu(t)$, we must calculate the velocity error coefficient ($\boldsymbol{K_v}$):

$$K_v = \lim_{s \to 0} sG(s) = \frac{100 \times 2 \times 6}{(3) \times (4)} = \frac{1200}{12} = 100$$

$$\boldsymbol{e_{ss} = \frac{5}{K_v} = \frac{5}{100} = 0.05}$$

For parabolic input $5t^2u(t)$, we must calculate the acceleration error coefficient ($\boldsymbol{K_a}$):

$$K_a = \lim_{s \to 0} s^2G(s) = \frac{(0) \times 100 \times 2 \times 6}{(3) \times (4)} = \frac{0}{12} = 0$$

$$\boldsymbol{e_{ss} = \frac{5 \times 2}{K_a} = \frac{10}{0} = \infty}$$

Assignment 3: Find the steady-state errors for inputs of $2u(t), 2tu(t)$, and $2t^2u(t)$ to the system shown below. The function $u(t)$ is the *unit step*.