Impulse Response of Second-Order Systems

For a unit-impulse input $r(t)$, the corresponding Laplace transform is unity, or $R(s) = 1$.The unit-impulse response $C(s)$ of the second-order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

The inverse Laplace transform of this equation yields the time solution for the response $c(t)$ as follows:

For $0 < \xi < 1$,

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \qquad \text{for } t \geq 0$$

For $\xi = 1$,

$$c(t) = \omega_n^2 t e^{-\omega_n t}, \qquad \text{for } t \geq 0$$

For $\xi > 1$,

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}, \qquad \text{for } t \geq 0$$

Note that without taking the inverse Laplace transform of $C(s)$ we can also obtain the time response $c(t)$ by differentiating the corresponding unit-step response, since the unit-impulse function is the time derivative of the unit-step function. A family of unit-impulse response curves with various values of ξ is shown in Figure (7 – 2). The curves $c(t)/\omega_n$ are plotted against the dimensionless variable $w_n t$, and thus they are functions only of ξ. For the critically damped and overdamped cases, the unit-impulse response is always positive or zero; that is, $c(t) \geq 0$. For the underdamped case, the unit-impulse response $c(t)$ oscillates about zero and takes both positive and negative values

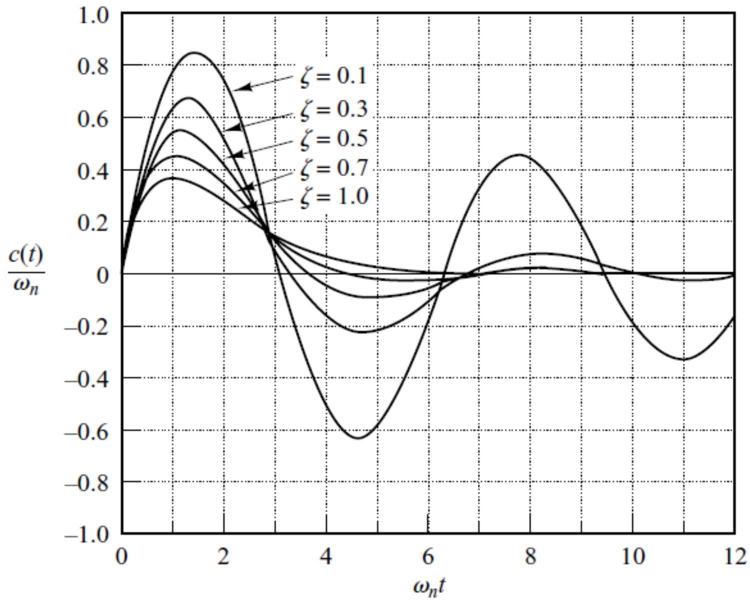

Figure (7 – 2): Unit-impulse response curves of the system shown in Figure 5–6.

From the foregoing analysis, we may conclude that if the impulse response $c(t)$ does not change sign, the system is either critically damped or overdamped, in which case the corresponding step response does not overshoot but increases or decreases monotonically and approaches a constant value.

The maximum overshoot for the unit-impulse response of the underdamped system occurs at

$$t = \frac{\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}}, \qquad \text{where } 0 < \zeta < 1$$

$$c(t)_{\max} = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{where } 0 < \zeta < 1$$

Since the unit-impulse response function is the time derivative of the unit-step response function, the maximum overshoot M_p for the unit-step response can be found from the corresponding unit-impulse response. That is, the area under the unit-impulse response curve from $t = 0$ to the time of the first zero, as shown in Figure (7 – 3), is $1 + M_p$, where M_p is the maximum overshoot (for the unit-step response). The peak time t_p (for the unit-step response) corresponds to the time that the unit-impulse response first crosses the time axis.

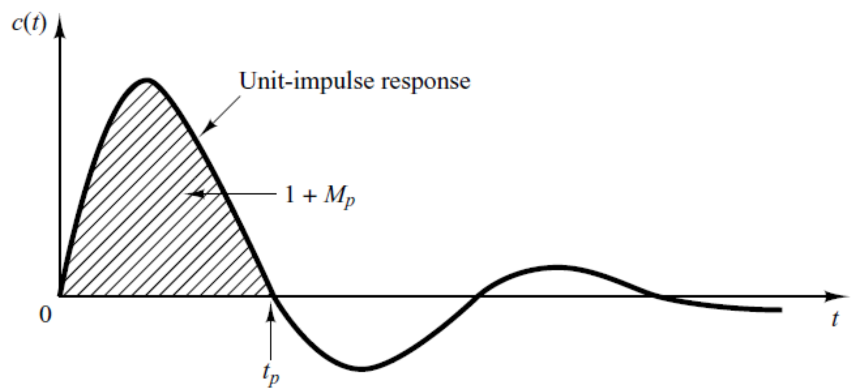

Figure (7 – 3): Unit-impulse response curve of the system

EFFECTS OF INTEGRAL AND DERIVATIVE CONTROL ACTIONS ON SYSTEM PERFORMANCE

we investigate the effects of integral and derivative control actions on the system performance. Integral Control Action. In the proportional control of a plant whose transfer function does not possess an integrator $1/s$, there is a steady-state error, or offset, in the response to a step input. Such an offset can be eliminated if the integral control action is included in the controller.

In the integral control of a plant, the control signal—the output signal from the controller—at any instant is the area under the actuating-error-signal curve up to that instant. The control signal $u(t)$ can have a nonzero value when the actuating error signal $e(t)$ is zero, as shown in Figure (7 3) -(a). This is impossible in the case of the proportional controller, since a nonzero control signal requires a nonzero actuating error signal.

(A nonzero actuating error signal at steady state means that there is an offset.) Figure (7 – 3)-(b) shows the curve $e(t)$ versus t and the corresponding curve $u(t)$ versus t when the controller is of the proportional type.

Note that integral control action, while removing offset or steady-state error, may lead to oscillatory response of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.

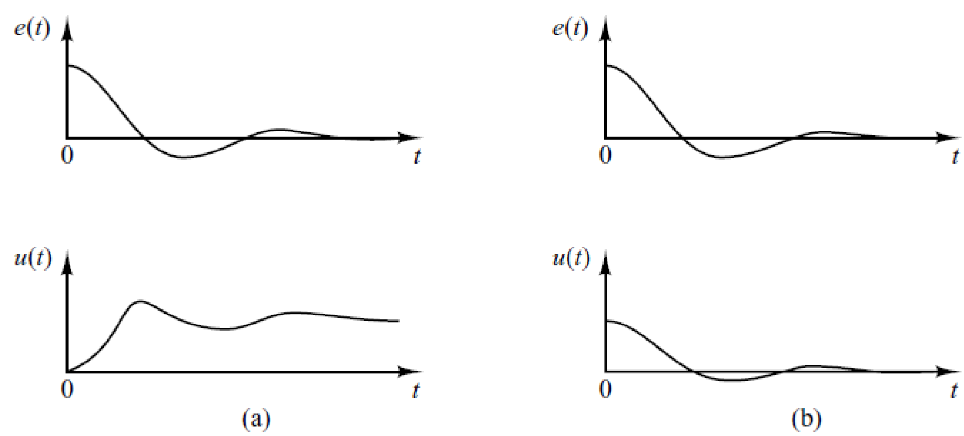

Figure (7 – 3): (a) Plots of $e(t)$ and $u(t)$ curves showing nonzero control signal when the actuating error signal is zero (integral control); (b) plots of $e(t)$ and $u(t)$ curves showing zero control signal when the actuating error signal is zero (proportional control).

Proportional Control of Systems

Consider the system shown in Figure (7–4). Let us obtain the steady-state error in the unit-step response of the system. Define

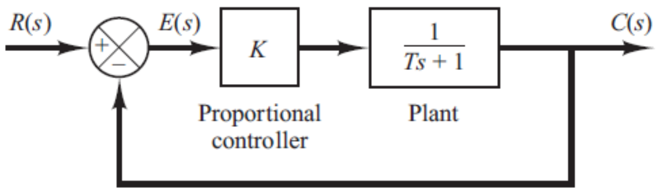

Figure (7–4): Plant Proportional control system.

$$G(s) = \frac{K}{Ts + 1}$$

Since

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$

the error $E(s)$ is given by

$$E(s) = \frac{1}{1 + G(s)} R(s) = \frac{1}{1 + \dfrac{K}{Ts + 1}} R(s)$$

For the unit-step input $R(s) = 1/s$, we have

$$E(s) = \frac{Ts + 1}{Ts + 1 + K} \frac{1}{s}$$

The steady-state error is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{Ts + 1}{Ts + 1 + K} = \frac{1}{K + 1}$$

Such a system without an integrator in the feedforward path always has a steady-state error in the step response. Such a steady-state error is called an offset. Figure (7–5) shows the unit-step response and the offset.

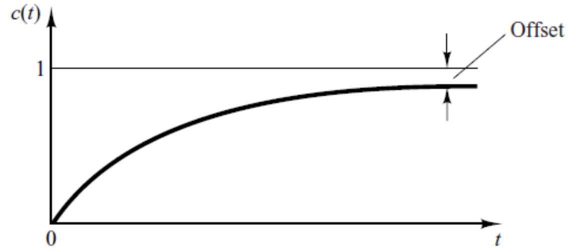

Figure (7–5): Unit-step response and offset.

Integral Control of Systems

Consider the system shown in Figure (7 – 6). The controller is an integral controller. The closed-loop transfer function of the system is

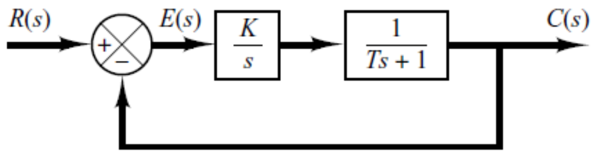

Figure (7 – 6): Integral control system.

$$\frac{C(s)}{R(s)} = \frac{K}{s(Ts + 1) + K}$$

Hence

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = \frac{s(Ts + 1)}{s(Ts + 1) + K}$$

Since the system is stable, the steady-state error for the unit-step response can be obtained by applying the final-value theorem, as follows:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{s^2(Ts + 1)}{Ts^2 + s + K} \frac{1}{s} \\ &= 0 \end{aligned}$$

Integral control of the system thus eliminates the steady-state error in the response to the step input. This is an important improvement over the proportional control alone, which gives offset.

PID Controllers:

PID controllers are found in a wide range of applications for industrial process control. Approximately 95% of the closed-loop operations of the industrial automation sector use PID controllers. PID stands for Proportional-Integral-Derivative. These three controllers are combined in such a way that it produces a control signal. As a feedback controller, it delivers the control output at desired levels. Before microprocessors were invented, PID control was implemented by the analog electronic components. But today all PID controllers are processed by the microprocessors. Programmable logic controllers also have the inbuilt PID controller instructions. Due to the flexibility and reliability of the PID controllers, these are traditionally used in process control applications. The term PID stands for proportional integral derivative and it is one kind of device used to control different process variables like pressure, flow, temperature, and speed in industrial applications. In this controller, a control loop feedback device is used to regulate all the process variables.

A closed-loop system like a PID controller includes a feedback control system. This system evaluates the feedback variable using a fixed point to generate an error signal. Based on that, it alters the system output. This procedure will continue till the error reaches Zero otherwise the value of the feedback variable becomes equivalent to a fixed point.

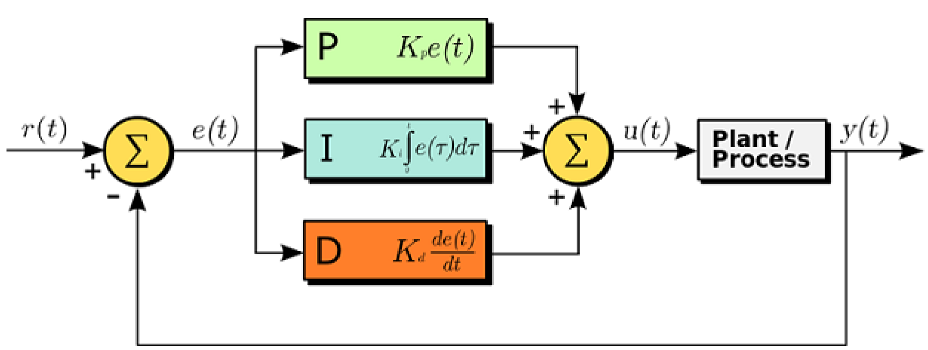

Figure (9 – 1): PID control system.

ZIEGLER–NICHOLS RULES FOR TUNING PID CONTROLLERS

PID Control of Plants.

Figure (9–2) shows a PID control of a plant. The process of selecting the controller parameters to meet given performance specifications is known as controller tuning. *Ziegler and Nichols* suggested rules for tuning PID controllers (meaning to set values $\boldsymbol{K_p, T_i}$ *and* $\boldsymbol{T_d}$) based on experimental step responses or based on the value of K_p that results in *marginal stability* when only proportional control action is used.

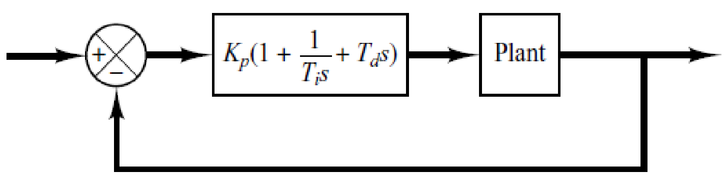

Figure (9 – 2): PID control of a plant.

Ziegler–Nichols rules, which are briefly presented in the following, are useful when mathematical models of plants are not known. (These rules can, of course, be applied to the design of systems with known mathematical models.) Such rules suggest a set of values of K_p, T_i *and* T_d that will give a *stable operation* of the system. However, the resulting system may exhibit a *large maximum overshoot* in the step response, which is unacceptable. In such a case we need series of fine tunings until an acceptable result is obtained. In fact, the Ziegler–Nichols tuning rules give an educated guess for the parameter values and provide a starting point for fine tuning, rather than giving the final settings for K_p, T_i *and* T_d in a single shot.

Note: If a mathematical model of the plant can be derived, then it is possible to apply various design techniques for determining parameters of the controller that will meet the transient and steady-state specifications of the closed-loop system. However, if the plant is so complicated that its mathematical model cannot be easily obtained, then an analytical or computational approach to the design of a PID controller is not possible. Then we must resort to experimental approaches to the tuning of PID controllers.

Ziegler–Nichols Rules for Tuning PID Controllers.

Ziegler and Nichols proposed rules for determining values of the proportional gain $\boldsymbol{K_p}$, integral time $\boldsymbol{T_i}$ and derivative time $\boldsymbol{T_d}$ based on the transient response characteristics of a given plant. Such determination of the parameters of PID controllers or tuning of PID controllers can be made by engineers on-site by experiments on the plant. (Numerous tuning rules for **PID** controllers have been proposed since the Ziegler–Nichols proposal. They are available in the literature and from the manufacturers of such controllers.) There are two methods called Ziegler–Nichols tuning rules: the first method and the second method. We shall give a brief presentation of these two methods.

First Method: In the first method, we obtain experimentally the response of the plant to a unit-step input, as shown in Figure (9–2). If the plant involves neither integrator (S) nor dominant complex-conjugate poles, then such a unit-step response curve may look S-shaped, as shown in Figure (9–3). This method applies if the response to a step input exhibits an $S - shaped\ curve$. Such step-response curves may be generated experimentally or from a dynamic simulation of the plant.

The $S - shaped\ curve$ may be characterized by two constants, delay time L and time constant T. The delay time and time constant are determined by drawing a tangent line at the inflection point of the $S - shaped\ curve$ and determining the intersections of the tangent line with the time axis and line $c(t) = K$, as shown in Figure (9–3).

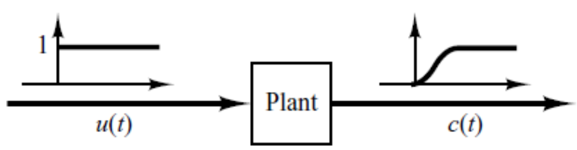

Figure (9–2): Unit-step response of a plant.

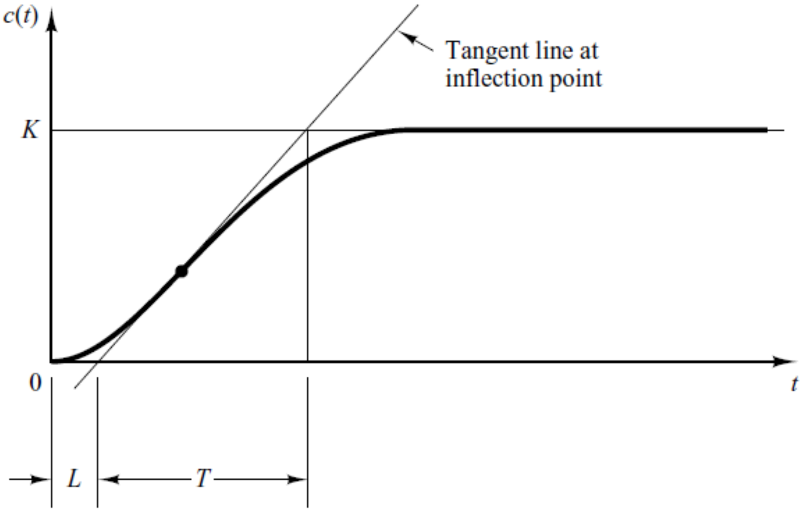

Figure (9–3): S-shaped response curve.

The transfer function $C(s)/U(s)$ may then be approximated by a first-order system with a transport lag as follows:

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts + 1}$$

Ziegler and Nichols suggested to set the values of and according to the formula shown in Table 9–1.

Table (9–1): Ziegler–Nichols Tuning Rule Based on Step Response of Plant (First Method)

Type of Controller	K_p	T_i	T_d
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

Notice that the PID controller tuned by the first method of Ziegler–Nichols rules gives

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$
$$= 1.2\frac{T}{L}\left(1 + \frac{1}{2Ls} + 0.5Ls\right)$$
$$= 0.6T\frac{\left(s + \frac{1}{L}\right)^2}{s}$$

Thus, the PID controller has a pole at the origin and double zeros at $s = -1/L$.

Second Method: In the second method, we first set and Using the proportional control action only (see Figure 9–4), increase $\boldsymbol{K_p}$ from 0 to a critical value $\boldsymbol{K_{cr}}$ at which the output first exhibits sustained oscillations. (If the output does not exhibit sustained oscillations for whatever value $\boldsymbol{K_p}$ may take, then this method does not apply.) Thus, the critical gain $\boldsymbol{K_{cr}}$ and the corresponding period $\boldsymbol{P_{cr}}$ are experimentally determined (see Figure 9–5). Ziegler and Nichols suggested that we set the values of the parameters $\boldsymbol{K_p}, \boldsymbol{T_i}$ *and* $\boldsymbol{T_d}$ according to the formula shown in Table 9–2.

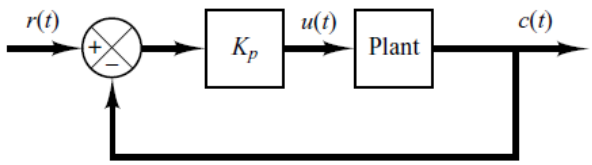

Figure (9–4): Closed-loop system with a proportional controller.

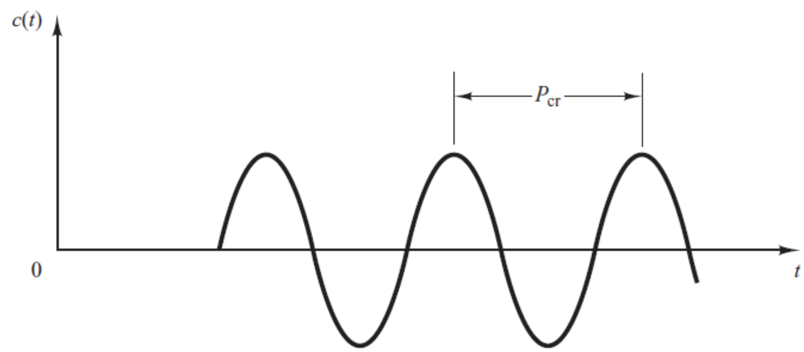

Figure (9–5): Sustained oscillation with period P_{cr} (P_{cr} is measured in sec.)

Table (9–2): Ziegler–Nichols Tuning Rule Based on Critical Gain K_{cr} and Critical Period P_{cr} (Second Method)

Type of Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

Notice that the PID controller tuned by the second method of Ziegler–Nichols rules gives

$$\begin{aligned} G_c(s) &= K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \\ &= 0.6K_{cr}\left(1 + \frac{1}{0.5P_{cr}s} + 0.125P_{cr}s\right) \\ &= 0.075K_{cr}P_{cr}\frac{\left(s + \frac{4}{P_{cr}}\right)^2}{s} \end{aligned}$$

Thus, the PID controller has a pole at the origin and double zeros at $s = -4/P_{cr}$.

Note that if the system has a known mathematical model (such as the transfer function), then we can use the root-locus method to find the critical gain P_{cr} and the frequency of the sustained oscillations ω_{cr}, where $2\pi/\omega_{cr} = P_{cr}$. These values can be found from the crossing points of the root-locus branches with the $j\omega$ axis. (Obviously, if the root-locus branches do not cross the $j\omega$ axis, this method does not apply.)

Example 1: Consider the control system shown in Figure 9–6 in which a PID controller is used to control the system. The PID controller has the transfer function

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

Although many analytical methods are available for the design of a PID controller for the present system, let us apply a Ziegler–Nichols tuning rule for the determination of the values of parameters $\boldsymbol{K_p, T_i}$ *and* $\boldsymbol{T_d}$. Then obtain a unit-step response curve and check to see if the designed system exhibits approximately 25% *maximum overshoot*. If the *maximum overshoot* is excessive (40% or more), make *a fine tuning and reduce the amount of the maximum overshoot* to approximately 25% or less.

Since the plant has an integrator, we use the second method of Ziegler–Nichols tuning rules.

By setting $T_i = \infty$ *and* $T_d = 0$, we obtain the closed-loop transfer function as follows:

$$\frac{C(s)}{R(s)} = \frac{K_p}{s(s+1)(s+5) + K_p}$$

The value of $\boldsymbol{K_p}$ that makes the system marginally stable so that sustained oscillation occurs can be obtained by use of *Routh's stability criterion*. Since the characteristic equation for the closed-loop system is

$$s^3 + 6s^2 + 5s + K_p = 0$$

the Routh array becomes as follows:

$$\begin{array}{lcc} s^3 & 1 & 5 \\ s^2 & 6 & K_p \\ s^1 & \dfrac{30 - K_p}{6} & \\ s^0 & K_p & \end{array}$$

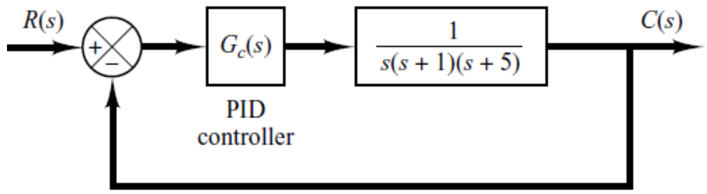

Figure (9–6): PID-controlled system.

Examining the coefficients of the first column of the Routh table, we find that sustained oscillation will occur if $K_p = 30$ Thus, the critical gain is K_{cr}

$$K_{cr} = 30$$

With gain K_p set equal to $K_{cr} = 30$ the characteristic equation becomes

$$s^3 + 6s^2 + 5s + 30 = 0$$

To find the frequency of the sustained oscillation, we substitute $s = j\omega$ into this characteristic equation as follows:

$$(j\omega)^3 + 6(j\omega)^2 + 5(j\omega) + 30 = 0$$

Or

$$6(5 - \omega^2) + j\omega(5 - \omega^2) = 0$$

from which we find the frequency of the sustained oscillation to be $\omega^2 = 5$ or $\omega = \sqrt{5}$.

Hence, the period of sustained oscillation is

$$P_{cr} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}} = 2.8099$$

Referring to Table 9–2, we determine $\boldsymbol{K_p}, \boldsymbol{T_i}$ *and* $\boldsymbol{T_d}$ as follows:

$$K_p = 0.6K_{cr} = 18$$

$$T_i = 0.5P_{cr} = 1.405$$

$$T_d = 0.125P_{cr} = 0.35124$$

The transfer function of the PID controller is thus

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$= 18\left(1 + \frac{1}{1.405s} + 0.35124s\right)$$

$$= \frac{6.3223(s + 1.4235)^2}{s}$$

The PID controller has a pole at the origin and double zero at $s = -1.4235$. A block diagram of the control system with the designed PID controller is shown in Figure 9–7.

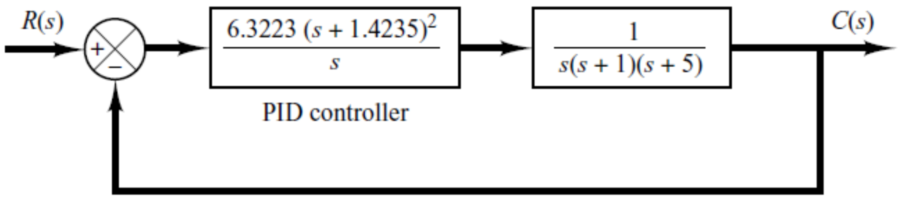

Figure (9–7): Block diagram of the system with PID controller designed by use of the Ziegler–Nichols tuning rule (second method).

Next, let us examine the unit-step response of the system. The closed-loop transfer function $C(s)/R(s)$ is given by

$$\frac{C(s)}{R(s)} = \frac{6.3223s^2 + 18s + 12.811}{s^4 + 6s^3 + 11.3223s^2 + 18s + 12.811}$$