Chapter nine

Complex numbers

If the imaginary unit i (where $i^2 = -1$) is combine with two real numbers α, β by the processes of addition and multiplication, we obtain a complex number $\alpha + i\beta$. If $\alpha = 0$, the number is said to be purely imaginary, if $\beta = 0$ it is of course real. Zero is the only number which is at once real and imaginary.

Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

i.e. $\alpha_1 + i\beta_1 = \alpha_2 + i\beta_2 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 \quad and \quad \beta_1 = \beta_2$

Assuming that the ordinary rules of arithmetic apply to complex numbers, we find indeed:-

1. $(\alpha_1 + i\beta_1) \mp (\alpha_2 + i\beta_2) = (\alpha_1 \mp \alpha_2) + i(\beta_1 \mp \beta_2)$
2. $(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)$
   where $i^2 = -1$
3. $\dfrac{\alpha_1 + i\beta_1}{\alpha_2 + i\beta_2} * \dfrac{\alpha_2 - i\beta_2}{\alpha_2 - i\beta_2} = \dfrac{\alpha_1\alpha_2 + \beta_1\beta_2}{\alpha_2^2 + \beta_2^2} + i\,\dfrac{\alpha_2\beta_1 + \alpha_1\beta_2}{\alpha_2^2 + i\beta_2^2}$

The real number $\alpha_2 - i\beta_2$ that is used as multiplier to clear the i out of the denominator is called the complex conjugate of $\alpha_2 + i\beta_2$. It is customary to use $\bar{z}$ to denote the complex conjugate of z, thus $z = \alpha + i\beta$ and $\bar{z} = \alpha - i\beta$.

We note that i^n has only four possible values $1 , i , -1 , -i$. They correspond to values of n which divided by 4 leave the reminders 0 , 1 , 2 , 3 .

<u>EX-1</u> **– Find the values of :**

$1)\ (1+2i)^3 \qquad 2)\ \dfrac{5}{-3+4i} \qquad 3)\ \left(\dfrac{2+i}{3-2i}\right)^2$

<u>Sol.</u> **–**

$$1)\ (1+2i)^3 = 1+6i+12i^2+8i^3 = 1+6i-12-8i = -11-2i$$

$$2)\ \frac{5}{-3+4i} * \frac{-3-4i}{-3-4i} = \frac{-15-20i}{9+16} = -\frac{3}{5} - i\frac{4}{5}$$

$$3)\ \left(\frac{2+i}{3-2i} * \frac{3+2i}{3+2i}\right)^2 = \left(\frac{6+7i-2}{9+4}\right)^2 = \left(\frac{4+7i}{13}\right)^2$$

$$= \frac{16+56i-49}{169} = -\frac{33}{169} + \frac{56}{169}i$$

EX-2- **If $z=x+iy$ where x and y are real, find the real and imaginary parts of:-**

$1)\ z^4 \qquad 2)\ \frac{1}{z} \qquad 3)\ \frac{z-1}{z+1} \qquad 4)\ \frac{1}{z^2}$

Sol.-

$$1)\ z^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$
$$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$$

$$2)\ \frac{1}{z} = \frac{1}{x+iy} * \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$3)\ \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} * \frac{(x+1)-iy}{(x+1)-iy} = \frac{x^2-1-2iy+y^2}{(x+1)^2+y^2}$$
$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} - i\frac{2y}{(x+1)^2+y^2}$$

$$4)\ \frac{1}{z^2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+2xyi} * \frac{x^2-y^2-2xyi}{x^2-y^2-2xyi}$$
$$= \frac{x^2-y^2-2xyi}{(x^2-y^2)^2+4x^2y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2}$$

EX-3- **Show that** $\left(\frac{-1 \mp i\sqrt{3}}{2}\right)^3 = 1$ **for all combination of signs.**

Sol.-

$$L.H.S.=\left(\frac{-1\mp i\sqrt{3}}{2}\right)^3=\frac{1}{8}\left[(-1)^3+3(-1)^2(\mp i\sqrt{3})+3(-1)(\mp i\sqrt{3})^2+(\mp i\sqrt{3})^3\right]$$

$$=\frac{1}{8}\left[-1\mp \cancel{i3\sqrt{3}}+9\pm \cancel{i3\sqrt{3}}\right]=1=R.H.S.$$

EX-4- **Solve the following equation for the real numbers x and y.**

$$(3+4i)^2-2(x-iy)=x+iy$$

Sol.-

$$9+24i+16i^2=2x-2iy+x+iy$$

$$-7+24i=3x-iy \;\Rightarrow\; -7=3x \;\Rightarrow\; x=-\frac{7}{3}$$

$$\hookrightarrow\; 24=-y \;\Rightarrow\; y=-24$$

Argrand Diagrams:- **There are two geometric representation of the complex number $z=x+iy$:-**

a) **as the point $P(x,y)$ in the xy-plane , and**
b) **as the vector $\overrightarrow{op}$ from the origin to P.**

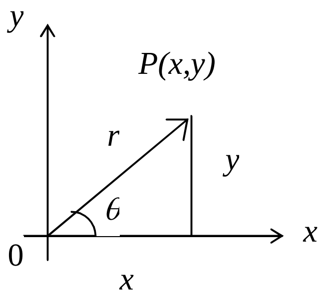
In each representation, the x-axis is called the real axis and the y-axis is the imaginary axis, as following figure.

In terms of the polar coordinates of x and y , we have:-

$$x=r\cos\theta \quad , \quad y=r\sin\theta \quad , \quad \tan\theta=\frac{y}{x}$$

and $z=r(\cos\theta+i\sin\theta)$

(polar representation)



The length r of a vector $\overrightarrow{op}$ from the origin to $P(x,y)$ is:

$$|x+iy|=\sqrt{x^2+y^2}$$

The polar angle θ is called the argument of z and is written $\theta=\arg z$

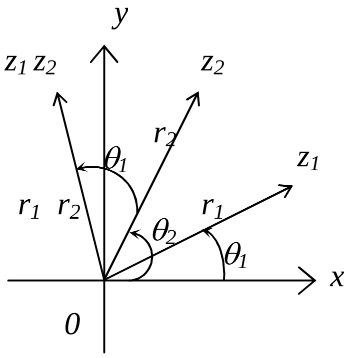
The identity $e^{i\theta} = \cos\theta + i\sin\theta$ is used for calculating products, quotients, powers, and roots of complex numbers. Then $z = re^{i\theta}$ exponential representation.

a) Product: To multiply two complex numbers (figure below):

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2} \quad \text{so that} \quad \begin{matrix} |z_1| = r_1 \quad , \quad \arg z_1 = \theta_1 \\ |z_2| = r_2 \quad , \quad \arg z_2 = \theta_2 \end{matrix}$$

Then $z_1\, z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1\, r_2\, e^{i(\theta_1+\theta_2)}$

hence $\begin{matrix} |z_1\, z_2| = r_1\, r_2 = |z_1| \cdot |z_2| \\ \arg(z_1\, z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2 \end{matrix}$



b) Quotients: $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

hence $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|} \quad \textit{and} \quad \arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

***EX-5*- Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ find:**

1) **the exponential representation for z_1 and z_2.**
2) **the values of $z_1\, z_2$ and $\dfrac{z_1}{z_2}$ in exponential and polar representations.**

***Sol.*-**

1) $z_1 = 1 + i \;\Rightarrow\; x_1 = 1 \;,\; y_1 = 1 \;\Rightarrow\; r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1+1} = \sqrt{2}$

$$\tan\theta = \frac{y}{x} \;\Rightarrow\; \theta_1 = \tan^{-1}\frac{y_1}{x_1} = \tan^{-1} 1 = \frac{\pi}{4} \qquad \therefore z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$z_2 = \sqrt{3} - i \;\Rightarrow\; x_2 = \sqrt{3} \;,\; y_2 = -1 \;\Rightarrow\; r_2 = \sqrt{x_2^2 + y_2^2} = \sqrt{3+1} = 2$$

$$\Rightarrow\; \theta_2 = \tan^{-1}\frac{y_2}{x_2} = \tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{\pi}{6} \qquad \therefore z_2 = 2\, e^{-i\frac{\pi}{6}}$$

$$2)\quad z_1 z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}} \cdot 2\, e^{-i\frac{\pi}{6}} = 2\sqrt{2} e^{i\frac{\pi}{12}} \qquad \text{exponential representation}$$

$$r = 2\sqrt{2} \quad , \quad \theta = \frac{\pi}{12} \qquad \Rightarrow$$

$$z_1 z_2 = 2\sqrt{2}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12} \right) \qquad \text{polar representation}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\frac{\pi}{4}}}{2\, e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}} e^{i\frac{5}{12}\pi} \qquad \text{exponential representation}$$

$$r = \frac{1}{\sqrt{2}} \quad , \quad \theta = \frac{5}{12}\pi \qquad \Rightarrow$$

$$\frac{z_1}{z_2} = \frac{1}{\sqrt{2}}\left(\cos(\frac{5}{12}\pi) + i \sin(\frac{5}{12}\pi) \right) \qquad \text{polar representation}$$

c) **Powers: If** n **is a positive integer, then:**

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \qquad \text{hence } \left| z^n \right| = r^n \quad \text{and} \quad \arg z^n = n\theta$$

$$\text{DeMoivres Theorem :} \quad (\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$$

***EX-6*-** **Find:** $\left(\sqrt{3} - i\right)^{10}$

Sol.-

$$\sqrt{3} - i \underset{y=-1}{\overset{x=\sqrt{3}}{\Longrightarrow}} \quad r = \sqrt{3+1} = 2 \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\sqrt{3} - i = 2\left(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}) \right) = 2\left(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6} \right)$$

$$\left(\sqrt{3} - i\right)^{10} = 2^{10}\left(\cos 10\frac{\pi}{6} - i \sin 10\frac{\pi}{6} \right) = 512 + 512\sqrt{3} i$$

d) **Roots: If** $z = re^{i\theta}$ **is a complex number different from zero and** n **is a positive integer, then there are precisely** n **different complex numbers** $w_0, w_1, w_2, \ldots\ldots, w_{n-1}$ **, that are *nth* roots of** z **given by:**

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)} \quad , \quad k = 0,1,2,\ldots\ldots\ldots, n-1$$

<u>*EX-7*</u>- Find the four forth roots of (-16)

<u>*Sol.*</u>-

$$z=-16 \Rightarrow r=\sqrt{(-16)^2+0}=16 \quad \& \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{0}{-16}=\pi$$

$$\sqrt[4]{-16}=\sqrt[4]{16}\, e^{i(\frac{\pi}{4}+k\frac{2\pi}{4})}=2e^{i(\frac{\pi}{4}+\frac{\pi}{2}k)} \qquad , \qquad k=0,1,2,3$$

$$\text{at } k=0 \Rightarrow \text{1st root } = w_0=2e^{i\frac{\pi}{4}}=2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}+\sqrt{2}i$$

$$\text{at } k=1 \Rightarrow \text{2nd root } = w_1=2e^{i\frac{3\pi}{4}}=2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-\sqrt{2}+\sqrt{2}i$$

$$\text{at } k=2 \Rightarrow \text{3rd root } = w_2=2e^{i\frac{5\pi}{4}}=2\left(\cos\frac{5}{4}\pi+i\sin\frac{5}{4}\pi\right)=-\sqrt{2}-\sqrt{2}i$$

$$\text{at } k=3 \Rightarrow \text{4th root } = w_3=2e^{i\frac{7}{4}\pi}=2\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)=\sqrt{2}-\sqrt{2}i$$

<u>*EX-8*</u>- Find the four solutions of the equation:- $z^4-2z^2+4=0$

<u>*Sol.*</u>-

$$z^4-2z^2+4=0 \Rightarrow z^2=\frac{2\mp\sqrt{4-4*1*4}}{2*1}=1\mp\sqrt{3}i \Rightarrow z=\mp\sqrt{1\mp i\sqrt{3}}$$

$$\left\{ \text{for} \quad ax^2+bx+c=0 \quad \Rightarrow \quad x=\frac{-b\mp\sqrt{b^2-4ac}}{2a} \right\}$$

$$\text{for } \sqrt{1+i\sqrt{3}} \Rightarrow r=\sqrt{1+3}=2 \quad \text{and} \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}$$

$$\text{1st root } = w_0=\sqrt{2}e^{\frac{i}{2}(\frac{\pi}{3})}=\sqrt{2}e^{i\frac{\pi}{6}}=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i$$

$$\text{2nd root } = w_1=\sqrt{2}e^{\frac{i}{2}(\frac{\pi}{3}+2\pi)}=\sqrt{2}e^{i\frac{7}{6}\pi}=\sqrt{2}\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$$

$$=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i$$

$$\text{for } \sqrt{1-i\sqrt{3}} \Rightarrow r=\sqrt{1+3}=2 \quad \text{and} \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{-\sqrt{3}}{1}=-\frac{\pi}{3}$$

$$3rd\ root = w_2 = \sqrt{2}e^{\frac{i}{2}(-\frac{\pi}{3})} = \sqrt{2}e^{i(-\frac{\pi}{6})} = \sqrt{2}\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)$$

$$= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

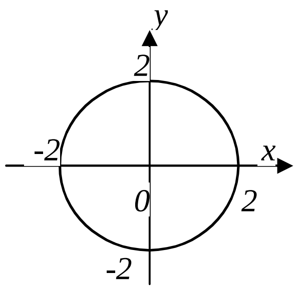
$$4th\ root = w_3 = \sqrt{2}e^{\frac{i}{2}(-\frac{\pi}{3}+2\pi)} = \sqrt{2}e^{i\frac{5}{6}\pi} = \sqrt{2}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right)$$

$$= -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

EX-9- **Graph the points $z = x + iy$ that satisfy the given conditions:-**

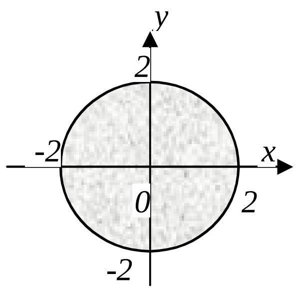
$1)\ |z| = 2 \qquad 2)\ |z| < 2 \qquad 3)\ |z| > 2 \qquad 4)\ |z+1| = |z-1|$

Sol.-



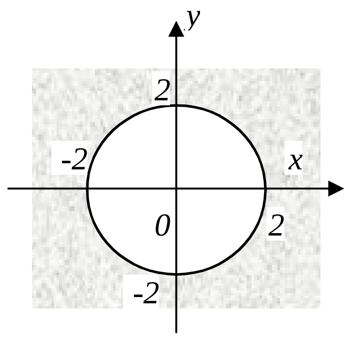
$1)\ |z| = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$

The points on the circle with center at origin, and radius 2.



$2)\ |z| < 2 \Rightarrow \sqrt{x^2 + y^2} < 2 \Rightarrow x^2 + y^2 < 4$

The interior points of the circle with center at origin, and radius 2.



$3)\ |z| > 2 \Rightarrow \sqrt{x^2 + y^2} > 2 \Rightarrow x^2 + y^2 > 4$

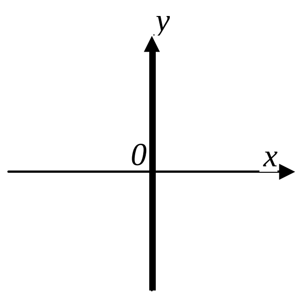
The exterior points of the circle with center at origin, and radius 2.

$4)\ |z+1| = |z-1| \Rightarrow |x + iy + 1| = |x + iy - 1|$

$\Rightarrow \sqrt{(x+1)^2 + y^2} = \sqrt{(x-1)^2 + y^2} \Rightarrow$

$x^2 + 2x + 1 + y^2 = x^2 - 2x + 1 + y^2 \Rightarrow x = 0$

The points on the y - axis.



Problems – 9

1) **Find the values of:-**

a) $(2+3i)(4-2i)$ (ans.: $14+8i$)

b) $(2-i)(-2+3i)$ (ans.: $-1+8i$)

c) $(-1-2i)(2+i)$ (ans.: $-5i$)

2) **Show that** $\left(\dfrac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$ **for all combination of signs.**

3) **Solve the following equation for the real numbers** x **and** y **:-**

$(3-2i)(x+iy) = 2(x-2iy) + 2i - 1$ (ans.: $x=-1$; $y=0$)

4) **Show that** $|\bar{z}| = |z|$.

5) **Let** $Re(z)$ **and** $Im(z)$ **denote respectively the real and imaginary parts of** z **, show that:-**

a) $z + \bar{z} = 2\,Re$

b) $z - \bar{z} = 2i\,Im(z)$

c) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1 \bar{z}_2)$

6) **Graph the points** $z = x + iy$ **that satisfy the given conditions:-**

a) $|z-1| = 2$ (ans.: on the circle with center (1,0), radius 2)

b) $|z+1| = 1$ (ans.: on the circle with center (−1,0), radius 1)

c) $|z+i| = |z-1|$ (ans.: on the line $y = -x$)

7) **Express the following complex number in exponential form with** $r \geq o$ *and* $-\pi < \theta < \pi$ **:-**

a) $(1+\sqrt{-3})^2$ (ans.: $4e^{i\frac{2}{3}\pi}$)

b) $\frac{1+i}{1-i}$ (ans.: $e^{i\frac{\pi}{2}}$)

c) $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ (ans.: $e^{i\frac{\pi}{2}}$)

d) $(2+3i)(1-2i)$ (ans.: $\sqrt{65}e^{i\tan^{-1}(-0.125)}$)

8) **Find the three cube roots of** 1 **.** (ans.: $-\frac{1}{2} \mp i\frac{\sqrt{3}}{2}$)

9) **Find the two square roots of** i **.** (ans.: $\mp\frac{1}{\sqrt{2}} \mp i\frac{1}{\sqrt{2}}$)

10) **Find the three cube roots of** *(-8i)* **.**

(ans.: $-2i$; $\mp\sqrt{3} - i$)

11) **Find the six sixth roots of** *(64)* **.**

(ans.: ∓ 2 ; $1 \mp i\sqrt{3}$; $-1 \mp i\sqrt{3}$)

12) **Find the six solutions of the equation:** $z^6 + 2z^3 + 2 = 0$

(ans.: $\sqrt[3]{2}\left(\cos\frac{2}{9}\pi \mp i\sin\frac{2}{9}\pi\right)$;

$\sqrt[3]{2}\left(-\cos\frac{\pi}{9} \mp i\sin\frac{\pi}{9}\right)$; $\sqrt[3]{2}\left(\cos\frac{4}{9}\pi \mp i\sin\frac{4}{9}\pi\right)$)

13) **Find all solutions of the equation:** $x^4 + 4z^2 + 16 = 0$

(ans.: $1 \mp i\sqrt{3}$; $-1 \mp i\sqrt{3}$)

14) **Solve the equation:** $x^4 + 1 = 0$

(ans.: $\frac{1}{\sqrt{2}} \mp \frac{i}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}} \mp \frac{i}{\sqrt{2}}$)