# Chapter – one
The Rate of Change of a Function

1-1- <u>Coordinates for the plane</u> :

<u>*Cartesian Coordinate*</u>**- Two number lines , one of them horizontal (called *x-axis*) and the other vertical (called *y-axis*). The point where the lines cross is the *origin* . Each line is assumed to represent the real number .**

On the *x-axis* , the positive number *a* lies *a* units to the right of the *origin* , and the negative number *–a* lies *a* units to the left of the *origin* . On the *y-axis* , the positive number *b* lies *b* units above the *origin* while the negative where *–b* lies *b* units below the *origin* .

With the axes in place , we assign a pair *(a,b)* of real number to each point *P* in the plane . The number *a* is the number at the foot of the perpendicular from *P* to the *x-axis* (called *x-coordinate of P*). The number *b* is the number at the foot of the perpendicular from *P* to the *y-axis* (called *y-coordinate of P*).
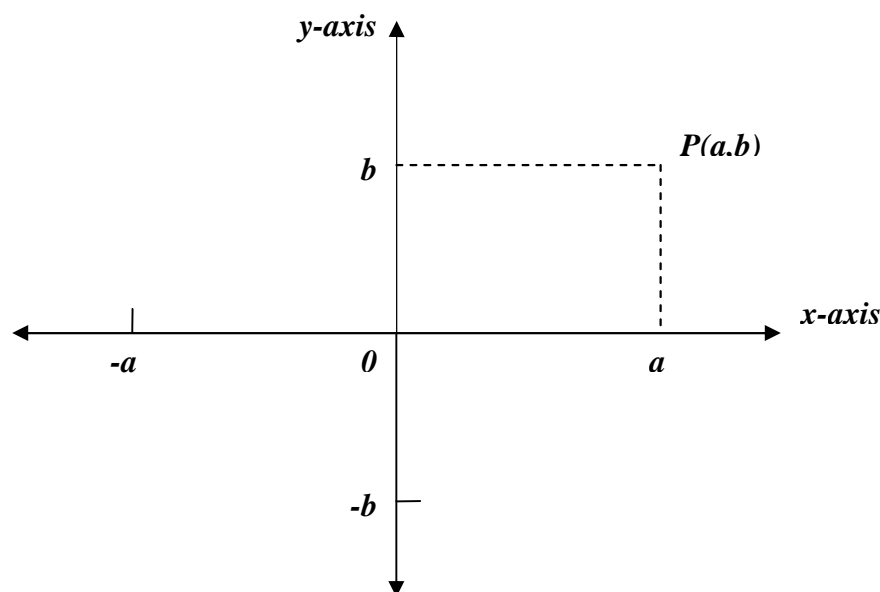
1-2- <u>The Slope of a line</u> :

<u>*Increments*</u> **– When a particle moves from one position in the plane to another , the net changes in the particle's coordinates are calculated by subtracting the coordinates of the starting point (x_1 , y_1) from the coordinates of the stopping point (x_2 , y_2) ,**

i.e. $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$.

<u>*Slopes of nonvertical lines*</u> **:**

Let *L* be a nonvertical line in the plane ,
Let $P_1(x_1 , y_1)$ and $P_2 (x_2 , y_2)$ be two points on *L*.
Then the slope *m* is :

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad where \quad \Delta x \neq 0$$

- A line that goes uphill as x increases has a positive slope . A line that goes downhill as x increases has a negative slope .
- A horizontal line has slope zero because $\Delta y = 0$.
- The slope of a vertical line is undefined because $\Delta x = 0$.
- Parallel lines have same slope .
- If neither of two perpendicular lines L_1 and L_2 is vertical , their slopes m_1 and m_2 are related by the equation : $m_1 . m_2 = -1$.

Angles of Inclination: The angle of inclination of a line that crosses the *x-axis* is the smallest angle we get when we measure counter clock from the *x-axis* around the point of intersection .

The slope of a line is the tangent of the line angle of inclination .

$$m = tan \, \Phi \quad where \, \Phi \, is \, the \, angle \, of \, inclination .$$

- The angle of inclination of a horizontal line is taken to be 0° .
- Parallel lines have equal angle of inclination .
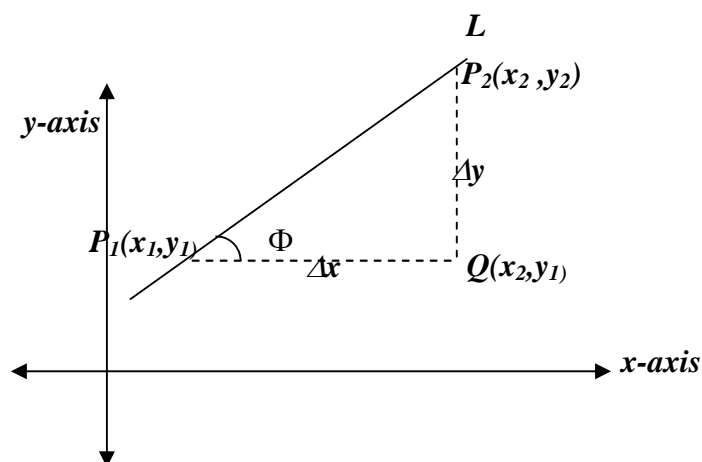
EX-1- Find the slope of the line determined by two points *A(2,1)* and *B(-1,3)* and find the common slope of the line perpendicular to *AB*.

Sol.- Slope of *AB* is: $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{-1 - 2} = -\frac{2}{3}$

Slope of line perpendicular to *AB* is : $-\frac{1}{m_{AB}} = \frac{3}{2}$

EX-2- Use slopes to determine in each case whether the points are collinear (lie on a common straight line) :
a) *A(1,0) , B(0,1) , C(2,1)* .
b) *A(-3,-2) , B(-2,0) , C(-1,2) , D(1,6)* .

<u>*Sol.*</u> –

a) $m_{AB} = \dfrac{1-0}{0-1} = -1 \quad and \quad m_{BC} = \dfrac{1-1}{2-0} = 0 \neq m_{AB}$

The points *A* , *B* and *C* are not lie on a common straight line .

b) $m_{AB} = \dfrac{0-(-2)}{-2-(-3)} = 2 \quad , \quad m_{BC} = \dfrac{2-0}{-1-(-2)} = 2 \quad , \quad m_{CD} = \dfrac{6-2}{1-(-1)} = 2$

Since $m_{AB} = m_{BC} = m_{CD}$

Hence the points *A* , *B* , *C* , and *D* are collinear .

1-3- <u>Equations for lines</u> : An equation for a line is an equation that is satisfied by the coordinates of the points that lies on the line and is not satisfied by the coordinates of the points that lie elsewhere .

<u>*Vertical lines*</u> **: Every vertical line *L* has to cross the x-axis at some point (a,0). The other points on *L* lie directly above or below (a,0) . This mean that : $x = a \quad \forall (x,y)$**

<u>*Nonvertical lines*</u> *:* **That point – slope equation of the line through the point (x_1 , y_1) with slope *m* is :**

$$y - y_1 = m (x - x_1)$$

<u>*Horizontal lines*</u> *:* **The standard equation for the horizontal line through the point (*a* , *b*) is : $y = b$.**

<u>*The distance from a point to a line*</u> *:* **To calculate the distance *d* between the point $P(x_1 , y_1)$ and $Q(x_2 , y_2)$ is :**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We use this formula when the coordinate axes are scaled in a common unit .

To find the distance from the point $P(x_1 , y_1)$ to the line *L* , we follow :

1. **Find an equation for the line *L′* through *P* perpendicular to *L* :**
   $$y - y_1 = m' (x - x_1) \quad \text{where } m' = -1 / m$$

2. **Find the point $Q(x_2 , y_2)$ by solving the equation for *L*   and   *L′* simultaneously .**

3. **Calculate the distance between *P* and *Q* .**

The general linear equation :

$$Ax + By = C \quad \text{where *A* and *B* not both zero.}$$

<u>*EX-3*</u> – **Write an equation for the line that passes through point :**

   a) **$P(-1 , 3)$ with slope $m = -2$.**

   b) **$P_1(-2 , 0)$ and $P_2(2 , -2)$.**

<u>*Sol.*</u> **-   a) $y - y_1 = m (x - x_1) \rightarrow y - 3 = -2 (x - (-1)) \rightarrow y + 2x = 1$**

   b)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-0}{2-(-2)} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 0 = -\frac{1}{2}(x - (-2)) \Rightarrow 2y + x + 2 = 0$$

EX-4 - Find the slope of the line : $3x + 4y = 12$.

Sol. - $y = -\dfrac{3}{4}x + 3 \Rightarrow$ the slope is $m = -\dfrac{3}{4}$

EX-5- Find :
   a) an equation for the line through $P(\ 2\ ,1\ )$ parallel to $L:\ \ y = x + 2$.
   b) an equation for the line through P perpendicular to L.
   c) the distance from P to L.

Sol.-

   a)
   $since\ \ \ L_2\ //\ L_1 \Rightarrow\ \ m_{L2} = m_{L1} = 1 \Rightarrow\ \ y - 1 = 1(\ x - 2\ )\ \ \Rightarrow\ \ y = x - 1$

   b) Since L_1 and L_3 are perpendicular lines then :
   $$m_{L3} = -1 \Rightarrow\ \ y - 1 = -(\ x - 2\ ) \Rightarrow\ \ y + x = 3$$

   c)
   $$\begin{array}{l} y = x + 2 \\ y + x = 3 \end{array} \Rightarrow\ \ x = \dfrac{1}{2}\ \ and\ \ \ y = \dfrac{5}{2}\ \Rightarrow\ \ P(2,1)\ \ and\ \ Q\left(\dfrac{1}{2},\dfrac{5}{2}\right)$$
   $$\Rightarrow d = \sqrt{(\ x_Q - x_P\ )^2 + (\ y_Q - y_P\ )^2} = \sqrt{4.5}$$

EX-6 – Find the angle of inclination of the line : $\ \sqrt{3}x + y = -3$

Sol.-
   $$y = -\sqrt{3}x - 3\ \ \ \Rightarrow\ \ m = -\sqrt{3}$$
   $$m = tan\ \Phi = -\sqrt{3}\ \ \Rightarrow\ \ \Phi = 120^o$$

EX-7- Find the line through the point $P(1,\ 4)$ with the angle of inclination
   $\Phi = 60^o$.

Sol.-
   $$m = tan\ \Phi = tan\ 60 = \sqrt{3}$$
   $$y - 4 = \sqrt{3}(\ x - 1\ ) \Rightarrow\ \ y = \sqrt{3}x + 4 - \sqrt{3}$$

EX-8- The pressure P experienced by a diver under water is related to the
   diver's depth d by an equation of the form $P = k\ d + 1$ where k a
   constant . When $d = 0$ meters , the pressure is 1 atmosphere . The
   pressure at 100 meters is about 10.94 atmosphere . Find the pressure
   at 50 meters.

Sol.- At $P = 10.94$ and $d = 100\ \ \ \rightarrow\ \ \ 10.94 = k(100) + 1 \rightarrow\ k = 0.0994$
   $P = 0.0994\ d + 1$, at $d = 50\ \ \rightarrow\ \ \ P = 0.0994 * 50 + 1 = 5.97$ atmo.

1-4- **Functions** : *Function* is any rule that assigns to each element in one set some element from another set :

$$y = f(x)$$

The inputs make up the *domain of the function* , and the outputs make up *the function's range.*

The variable x is called *independent variable of the function* , and the variable y whose value depends on x is called *the dependent variable of the function* .

We must keep two restrictions in mind when we define functions :
1. We never divide by zero .
2. We will deal with real – valued functions only.

Intervals :
- The *open interval* is the set of all real numbers that be strictly between two fixed numbers a and b :

$$(a,b) \equiv a < x < b$$

- The *closed interval* is the set of all real numbers that contain both endpoints :

$$[a,b] \equiv a \leq x \leq b$$

- *Half open interval* is the set of all real numbers that contain one endpoint but not both :

$$[a,b) \equiv a \leq x < b$$

$$(a,b] \equiv a < x \leq b$$

Composition of functions : suppose that the outputs of a function f can be used as inputs of a function g . We can then hook f and g together to form a new function whose inputs are the inputs of f and whose outputs are the numbers :

$$(g_O f)(x) = g(f(x))$$

EX-9- Find the domain and range of each function :

$$a) \quad y = \sqrt{x + 4} \quad , \quad b) \quad y = \frac{1}{x - 2}$$

$$c) \quad y = \sqrt{9 - x^2} \quad , \quad d) \quad y = \sqrt{2 - \sqrt{x}}$$

Sol. - $a)$ $\quad x + 4 \geq 0 \Rightarrow \quad x \geq -4 \Rightarrow \quad D_x : \forall x \geq -4 \quad , \quad R_y : \forall y \geq 0$

$\quad b)$ $\quad x - 2 \neq 0 \Rightarrow \quad x \neq 2 \Rightarrow \quad D_x : \forall x \neq 2$

$$y = \frac{1}{x - 2} \Rightarrow \quad x = \frac{1}{y} + 2 \Rightarrow \quad R_y : \forall y \neq 0$$

$\quad c)$ $\quad 9 - x^2 \geq 0 \Rightarrow \quad -3 \leq x \leq 3 \Rightarrow \quad D_x : -3 \leq x \leq 3$

$$y = \sqrt{9 - x^2} \Rightarrow \quad x = \mp\sqrt{9 - y^2}$$

$\quad \sin ce \quad 9 - y^2 \geq 0 \Rightarrow \quad -3 \leq y \leq 3$

$\quad \sin ce \quad y \geq 0 \quad\quad\quad\quad \Rightarrow \quad\quad R_y : 0 \leq y \leq 3$

$$d) \qquad 2 - \sqrt{x} \geq 0 \Rightarrow \qquad 0 \leq x \leq 4 \Rightarrow \quad D_x : 0 \leq x \leq 4$$
$$if \quad x = 0 \Rightarrow \quad y = \sqrt{2}$$
$$\Rightarrow \qquad R_y : 0 \leq y \leq \sqrt{2}$$
$$if \quad x = 4 \Rightarrow \quad y = 0$$

EX-10- Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = 1 + \dfrac{1}{x}$.

Find $(g_o f)(x)$ and $(f_o g)(x)$.

Sol.-

$$(g_o f)(x) = g(f(x)) = g\left(\frac{x}{x-1}\right) = 1 + \frac{1}{\dfrac{x}{x-1}} = \frac{2x-1}{x}$$

$$(f_o g)(x) = f(g(x)) = f\left(1 + \frac{1}{x}\right) = \frac{1 + \dfrac{1}{x}}{1 + \dfrac{1}{x} - 1} = x + 1$$

EX-11- Let $(g_o f)(x) = x$ and $f(x) = \dfrac{1}{x}$. Find $g(x)$.

Sol.- $(g_o f)(x) = g\left(\dfrac{1}{x}\right) = x \Rightarrow g(x) = \dfrac{1}{x}$

1-5- Limits and continuity :

Limits : The limit of $F(t)$ as t approaches C is the number L if :

Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about C such that for all t , $0 < |t - C| < \delta$ implies $|F(t) - L| < \varepsilon$ and we can write it as :

$$\lim_{t \to C} F(t) = L$$

The limit of a function $F(t)$ as $t \to C$ never depend on what happens when $t = C$.

Right hand limit : $\lim_{t \to C^+} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the right equals L if :
Given any $\varepsilon > 0$ (radius about L) there exists a $\delta > 0$ (radius to the right of C) such that for all t :

$$C < t < C + \delta \Rightarrow \quad |F(t) - L| < \varepsilon$$

Left hand limit : $\lim_{t \to C^-} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the left equal L if :
Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all t :

$$C - \delta < t < C \Rightarrow \quad |F(t) - L| < \varepsilon$$

Note that – A function $F(t)$ has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols :

$$\lim_{t \to C} F(t) = L \iff \lim_{t \to C^+} F(t) = L \quad and \quad \lim_{t \to C^-} F(t) = L$$

The limit combinations theorems :

1) $\lim[F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$

2) $\lim[F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$

3) $\lim \dfrac{F_1(t)}{F_2(t)} = \dfrac{\lim F_1(t)}{\lim F_2(t)} \quad$ *where* $\quad \lim F_2(t) \neq 0$

4) $\lim[k * F_1(t)] = k * \lim F_1(t) \quad \forall k$

5) $\lim_{\theta \to 0} \dfrac{Sin\,\theta}{\theta} = 1$

provided   that θ is   measured   in   radius

The limits (in *1 – 4*) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

Thm. -1 : The sandwich theorem *:* **Suppose that** $f(t) \leq g(t) \leq h(t)$ **for all** $t \neq C$ **in some interval about** C **and that** $f(t)$ **and** $h(t)$ **approaches the same limit** L **as** $t \to C$ **, then :**

$$\lim_{t \to C} g(t) = L$$

Infinity as a limit :

1.The limit of the function $f(x)$ as x approaches infinity is the number L:
$\lim_{x \to \infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number M such that
for all $x : M < x \implies |f(x) - L| < \varepsilon$.

2. The limit of $f(x)$ as x approaches negative infinity is the number L :
$\lim_{x \to -\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number N such that
for all $x : x < N \implies |f(x) - L| < \varepsilon$.

The following facts are some times abbreviated by saying :

    a) As x approaches 0 from the right , $1/x$ tends to ∞ .
    b) As x approaches 0 from the left , $1/x$ tends to $-\infty$.
    c) As x tends to ∞ , $1/x$ approaches 0 .
    d) As x tends to $-\infty$, $1/x$ approaches 0 .

Continuity :

Continuity at an interior point : A function $y = f(x)$ is continuous at an interior point C of its domain if : $\lim_{x \to C} f(x) = f(C)$.

Continuity at an endpoint : A function $y = f(x)$ is continuous at a left endpoint a of its domain if : $\lim_{x \to a^+} f(x) = f(a)$.

A function $y = f(x)$ is continuous at a right endpoint b of its domain if: $\lim_{t \to b^-} f(x) = f(b)$.

Continuous function : **A function is continuous if it is continuous at each point of its domain .**

Discontinuity at a point : **If a function f is not continuous at a point C , we say that f is discontinuous at C , and call C a point of discontinuity of f .**

The continuity test : **The function $y = f(x)$ is continuous at $x = C$ if and only if all three of the following statements are true :**

1) $f(C)$ **exist (C is in the domain of f) .**
2) $\lim\limits_{x \to C} f(x)$ **exists (f has a limit as $x \to C$) .**
3) $\lim\limits_{x \to C} f(x) = f(C)$ **(the limit equals the function value) .**

Thm.-2 : *The limit combination theorem for continuous function :*
   If the function f and g are continuous at $x = C$, then all of the following combinations are continuous at $x = C$:

$$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g \ \forall k \quad 4)\ g_\circ f\ ,\ f_\circ g \quad 5)\ f/g$$

$$provided \quad g(C) \neq 0$$

Thm.-3 : **A function is continuous at every point at which it has a derivative . That is , if $y = f(x)$ has a derivative $f'(C)$ at $x = C$, then f is continuous at $x = C$.**

EX-12 – Find :

$$1)\quad \lim_{x \to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} \qquad ,\quad 2)\quad \lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4}$$

$$3)\quad \lim_{x \to 0} \frac{Sin5x}{Sin3x} \qquad ,\quad 4)\quad \lim_{y \to 0} \frac{tan2y}{3y}$$

$$5)\quad \lim_{x \to 0} \frac{Sin2x}{2x^2 + x} \qquad ,\quad 6)\quad \lim_{x \to \infty} \left(1 + Cos\frac{1}{x}\right)$$

$$7)\quad \lim_{x \to \infty} \frac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5} \quad ,\quad 8)\quad \lim_{y \to \infty} \frac{3y + 7}{y^2 - 2}$$

$$9)\quad \lim_{x \to \infty} \frac{x^3 - 1}{2x^2 - 7x + 5} \quad ,\quad 10)\quad \lim_{x \to -1^-} \frac{1}{x + 1}$$

$$11)\quad \lim_{x \to 0} Cos\left(1 - \frac{Sinx}{x}\right) \quad ,\quad 12)\quad \lim_{x \to 0} Sin\left(\frac{\pi}{2} Cos(tan\,x)\right)$$

S0l.-

$$1)\quad \lim_{x \to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \lim_{x \to 0} \frac{5x + 8}{3x^2 - 16} = \frac{0 + 8}{0 - 16} = -\frac{1}{2}$$

$$2)\quad \lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4} = \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2)}{(x - a)(x + a)(x^2 + a^2)} = \frac{a^2 + a^2 + a^2}{(a + a)(a^2 + a^2)} = \frac{3}{4a}$$

$$3)\quad \lim_{x \to 0} \frac{5\dfrac{Sin5x}{5x}}{3\dfrac{Sin3x}{3x}} = \frac{5}{3} \cdot \frac{\lim\limits_{5x \to 0} \dfrac{Sin5x}{5x}}{\lim\limits_{3x \to 0} \dfrac{Sin3x}{3x}} = \frac{5}{3}$$

4) $\lim\limits_{y\to 0}\dfrac{tan\,2\,y}{3\,y}=\dfrac{2}{3}\cdot\lim\limits_{2\,y\to 0}\dfrac{Sin\,2\,y}{2\,y}\cdot\lim\limits_{y\to 0}\dfrac{1}{Cos\,2\,y}=\dfrac{2}{3}$

5) $\lim\limits_{x\to 0}\dfrac{Sin\,2\,x}{2\,x^{2}+x}=2\lim\limits_{2\,x\to 0}\dfrac{Sin\,2\,x}{2\,x}\cdot\lim\limits_{x\to 0}\dfrac{1}{2\,x+1}=2$

6) $\lim\limits_{x\to\infty}\left(1+Cos\dfrac{1}{x}\right)=1+Cos\,0=2$

7) $\lim\limits_{x\to\infty}\dfrac{3\,x^{3}+5\,x^{2}-7}{10\,x^{3}-11\,x+5}=\lim\limits_{x\to\infty}\dfrac{3+\dfrac{5}{x}-\dfrac{7}{x^{3}}}{10-\dfrac{11}{x^{2}}+\dfrac{5}{x^{3}}}=\dfrac{3}{10}$

8) $\lim\limits_{y\to\infty}\dfrac{3\,y+7}{y^{2}-2}=\lim\limits_{y\to\infty}\dfrac{\dfrac{3}{y}+\dfrac{7}{y^{2}}}{1-\dfrac{2}{y^{2}}}=\dfrac{0}{1}=0$

9) $\lim\limits_{x\to\infty}\dfrac{x^{3}-1}{2\,x^{2}-7\,x+5}=\lim\limits_{x\to\infty}\dfrac{1-\dfrac{1}{x^{3}}}{\dfrac{2}{x}-\dfrac{7}{x^{2}}+\dfrac{5}{x^{3}}}=\dfrac{1}{0}=\infty$

10) $\lim\limits_{x\to -1^{-}}\dfrac{1}{x+1}=\dfrac{1}{-1+1}=-\infty$

11) $\lim\limits_{x\to 0}Cos\left(1-\dfrac{Sinx}{x}\right)=Cos\left(1-\lim\limits_{x\to 0}\dfrac{Sinx}{x}\right)=Cos\,0=1$

12) $\lim\limits_{x\to 0}Sin\left(\dfrac{\pi}{2}Cos(tan\,x)\right)=Sin\left(\dfrac{\pi}{2}Cos(tan\,0)\right)=Sin\left(\dfrac{\pi}{2}Cos\,0\right)=Sin\dfrac{\pi}{2}=1$

EX-13- **Test continuity for the following function :**

$$f(x)=\begin{cases} x^{2}-1 & -1\le x<0\\ 2x & 0\le x<1\\ 1 & x=1\\ -2x+4 & 1<x\le 2\\ 0 & 2<x\le 3 \end{cases}$$

Sol.- We test the continuity at midpoints $x=0$ **,** 1 **,** 2 **and endpoints** $x=-1$ **,** 3 **.**

At $x=0\Rightarrow\qquad f(0)=2*0=0$

$$\lim\limits_{x\to 0^{-}}f(x)=\lim\limits_{x\to 0}(x^{2}-1)=-1$$

$$\lim\limits_{x\to 0^{+}}f(x)=\lim\limits_{x\to 0}2x=0\ne\lim\limits_{x\to 0^{-}}f(x)$$

Since $\lim\limits_{x\to 0}f(x)$ doesn't exist

Hence the function discontinuous at $x=0$

At   x = 1 ⟹   f(1) = 1

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 2x = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1}(-2x + 4) = 2 = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x)$$

Since $\lim_{x \to 1} f(x) \neq f(1)$

Hence   the function is   discontinuous   at   x = 1

*At   x = 2 ⟹   f(2) = −2 * 2 + 4 = 0*

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2}(-2x + 4) = 0$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} 0 = 0 = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x)$$

Since $\lim_{x \to 2} f(x) = f(2) = 0$

Hence   the function is   continuous   at   x = 2

At   x = −1 ⟹   f(−1) = (−1)² − 1 = 0

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1}(x^2 - 1) = 0 = f(-1)$$

Hence   the function is   continuous   at   x = −1

At   x = 3 ⟹   f(3) = 0

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} 0 = 0 = f(3)$$

Hence   the function is   continuous   at   x = 3

_**EX-14-**_ **What value should be assigned to *a* to make the function :**

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases} \quad \textbf{continuous at } x = 3 \text{ ?}$$

**Sol.** –

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) \Rightarrow \lim_{x \to 3}(x^2 - 1) = \lim_{x \to 3} 2ax \Rightarrow 8 = 6a \Rightarrow a = \frac{4}{3}$$

Problems – 1

1. The steel in railroad track expands when heated . For the track temperature encountered in normal outdoor use , the length S of a piece of track is related to its temperature t by a linear equation . An experiment with a piece of track gave the following measurements :

$$t_1 = 65^o F \quad , \quad S_1 = 35 \ ft$$

$$t_2 = 135^o F \quad , \quad S_2 = 35.16 \ ft$$

Write a linear equation for the relation between S and t.

(ans.: S=0.0023t+34.85)

2. Three of the following four points lie on a circle center the origin . Which are they , and what is the radius of the circle ?

A(-1.7) , B(5,-5) , C(-7,5) and D(7,-1). *(ans.: A,B,D;√50)*

3. *A* and *B* are the points *(3,4)* and *(7,1)* respectively . Use Pythagoras theorem to prove that *OA* is perpendicular to *AB* . Calculate the slopes of *OA* and *AB* , and find their product . *(ans.: 4/3, -3/4;-1)*

4. *P(-2,-4)* , *Q(-5,-2)* , *R(2,1)* and *S* are the vertices of a parallelogram . Find the coordinates of *M* , the point of intersection of the diagonals and of *S*.

(ans.: M(0,-3/2) , S(5,-1))

5. Calculate the area of the triangle formed by the line *3x-7y+4 =0* , and the axes . *(ans.: 8/21)*

6. Find the equation of the straight line through *P(7,5)* perpendicular to the straight line *AB* whose equation is *3x + 4y -16 = 0* . Calculate the length of the perpendicular from *P* and *AB*. *(ans.: 3y-4x+13=0;5)*

7. *L(-1,0)* , *M(3,7)* and *N(5,-2)* are the mid-points of the sides *BC* , *CA* and *AB* respectively of the triangle *ABC*. Find the equation of *AB*. *(ans.:4y=7x-43)*

8. The straight line *x – y – 6 = 0* cuts the curve $y^2 = 8x$ at *P* and *Q* . Calculate the length of *PQ* . *(ans.:16√2)*

9. A line is drawn through the point *(2,3)* making an angle of *45°* with the positive direction of the x-axis and it meets the line *x = 6* at *P* . Find the distance of *P* from the origin *O* , and the equation of the line through *P* perpendicular to *OP*. *(ans.: √85,7y+6x-85=0)*

10. The vertices of a quadrilateral *ABCD* are A(4,0) , B(14,11) , C(0,6) and D(-10,-5) . Prove that the diagonals *AC* and *BD* bisect each other at right angles , and that the length of *BD* is four times that of *AC* .

11. The coordinates of the vertices A, B and C of the triangle ABC are *(-3,7)* , *(2,19)* and *(10,7)* respectively :
   a) Prove that the triangle is isosceles.
   b) Calculate the length of the perpendicular from B to AC , and use it to find the area of the triangle . *(ans.:12,78)*

12. Find the equations of the lines which pass through the point of intersection of the lines $x - 3y = 4$ and $3x + y = 2$ and are respectively parallel and perpendicular to the line $3x + 4y = 0$.
(ans.:4y+3x+1=0;3y-4x+7=0)

13. Through the point $A(1,5)$ is drawn a line parallel to the x-axis to meet at B the line PQ whose equation is $3y = 2x - 5$. Find the length of AB and the sine of the angle between PQ and AB ; hence show that the length of the perpendicular from A to PQ is $18/\sqrt{13}$. Calculate the area of the triangle formed by PQ and the axes . *(ans.:9,2/√13,25/12)*

14. Let $y = \dfrac{x^2 + 2}{x^2 - 1}$, **express** x **in terms of** y **and find the values of** y **for**
   which x **is real .** $\quad$ (ans.: $x = \mp\sqrt{\dfrac{y+2}{y-1}}$; $y \le -2$ or $y > 1$)

15. Find the domain and range of each function :
$$a)\ y = \frac{1}{1+x^2} \quad,\quad b)\ y = \frac{1}{1+\sqrt{x}} \quad,\quad c)\ y = \frac{1}{\sqrt{3-x}}$$
$(ans.: a)\forall x, 0 < y \le 1;\ b)x \ge 0, y > 0;\ c)x < 3, y > 0)$

16. Find the points of intersection of $x^2 = 4y$ and $y = 4x$. *(ans.:(0,0),(16,64))*

17. Find the coordinates of the points at which the curves cut the axes :
$a)\ y = x^3 - 9x^2 \quad,\quad b)\ y = (x^2 - 1)(x^2 - 9) \quad,\quad c)\ y = (x+1)(x-5)^2$
$(ans.:a)(0,0);(0,0),(9,0);b)(0,9);(1,0),(-1,0),(3,0),(-3,0);c)(0,25);(-1,0),(5,0))$

18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a , b , c and d to make $f(g(x))$ and $g(f(x))$ identical ?
(ans.:ad+b=bc+d)

19. A particle moves in the plane from *(-2,5)* to the y-axis in such away that $\Delta y = 3 * \Delta x$. Find its new coordinates . *(ans.:(0,11),(0,-1))*

20. If $f(x) = 1/x$ and $g(x) = 1/\sqrt{x}$, what are the domain of f , g , $f+g$, $f-g$, $f.g$, f/g , g/f , $f_o g$ and $g_o f$? What is the domain of $h(x) = g(x+4)$?
$(ans.: \forall x \ne 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x \ge 0, \forall x \ge 0, \forall x \ge 0; \forall x > -4)$

21. Discuss the continuity of :

$$f(x) = \begin{cases} x + \dfrac{1}{x} & for & x < 0 \\ -x^3 & for & 0 \le x < 1 \\ -1 & for & 1 \le x < 2 \\ 1 & for & x = 2 \\ 0 & for & x > 2 \end{cases}$$

(ans.: discontinuous at x=0,2 ; continuous at x=1)

22. Evaluate the following limits :

$a)\ \lim\limits_{x \to \infty} \dfrac{x + Sinx}{2x + 5}$ $b)\ \lim\limits_{x \to \infty} \dfrac{1 + Sinx}{x}$

$c)\ \lim\limits_{x \to 0} \dfrac{x}{tan\,3x}$ $d)\ \lim\limits_{x \to \infty} \dfrac{x.Sinx}{(x + Sinx)^2}$

$e)\ \lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x}$ $f)\ \lim\limits_{x \to 1} \dfrac{\sqrt{x + 1} - \sqrt{2x}}{x^2 - x}$

$g)\ \lim\limits_{n \to \infty} (\sqrt{n^2 + 1} - n)$

(ans.:a)1/2, b)0, c)1/3, d)0, e)1/2, f)-1/2√2, g)0)

23. Suppose that : $f(x) = x^3 - 3x^2 - 4x + 12$ **and** $h(x) = \begin{cases} \dfrac{f(x)}{x - 3} & for\ x \ne 3 \\ k & for\ x = 3 \end{cases}$.

Find : a) all zeros of f **.**

b) the value of k **that makes** h **continuous at** $x=3$ **.**

$(ans.: a)x = \mp 2,3 ; b)k = 5)$