

Chapter two Functions

2-1- Exponential and Logarithm functions :

Exponential functions : If a is a positive number and x is any number , we define the exponential function as :

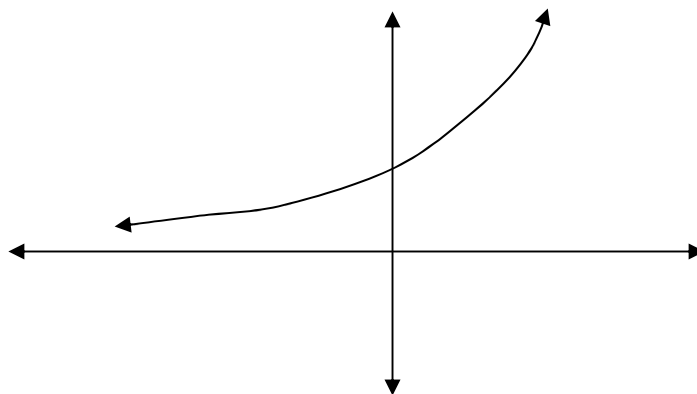
$$y = a^x \quad \text{with domain : } -\infty < x < \infty$$

$$\text{Range : } y > 0$$

The properties of the exponential functions are :

1. If $a > 0 \leftrightarrow a^x > 0$.
2. $a^x \cdot a^y = a^{x+y}$.
3. $a^x / a^y = a^{x-y}$.
4. $(a^x)^y = a^{x \cdot y}$.
5. $(a \cdot b)^x = a^x \cdot b^x$.
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$.
7. $a^{-x} = 1 / a^x$ and $a^x = 1 / a^{-x}$.
8. $a^x = a^y \leftrightarrow x = y$.
9. $a^0 = 1$,
 $a^\infty = \infty$, $a^{-\infty} = 0$, where $a > 1$.
 $a^\infty = 0$, $a^{-\infty} = \infty$, where $a < 1$.

The graph of the exponential function $y = a^x$ is :



Logarithm function : If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :

$$y = \log_a x \quad \text{where } x > 0$$

At $a = e = 2.7182828\dots$, we get the natural logarithm and denoted by :

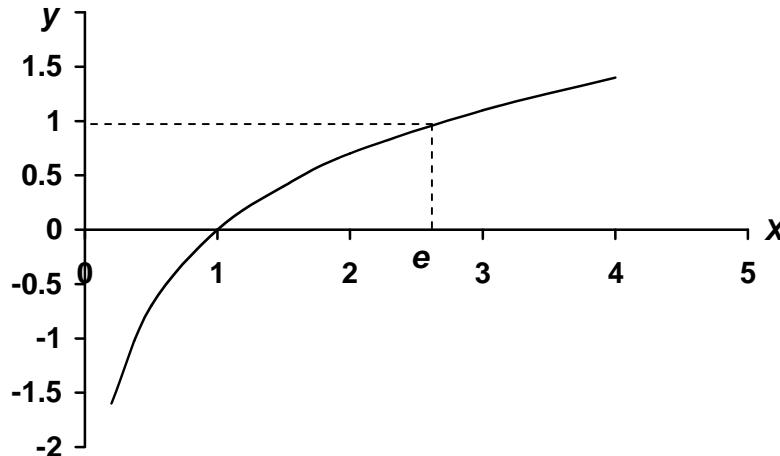
$$y = \ln x$$

Let $x, y > 0$ then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x. \ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = \ln x$ is :



Application of exponential and logarithm functions :

We take Newton's law of cooling :

$$T - T_s = (T_0 - T_s) e^{tk}$$

where T is the temperature of the object at time t .

T_s is the surrounding temperature .

T_0 is the initial temperature of the object .

k is a constant .

EX-1- The temperature of an ingot of metal is $80^\circ C$ and the room temperature is $20^\circ C$. After twenty minutes, it was $70^\circ C$.

- a) What is the temperature will the metal be after 30 minutes?
- b) What is the temperature will the metal be after two hours?
- c) When will the metal be $30^\circ C$?

Sol. :

$$T - T_s = (T_0 - T_s) e^{tk} \Rightarrow 50 = 60 e^{20k} \Rightarrow k = \frac{\ln 5 - \ln 6}{20} = -0.0091$$

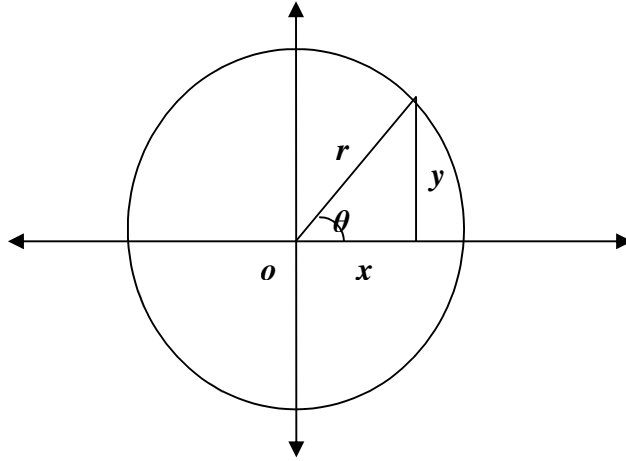
$$a) \quad T - 20 = 60 e^{30(-0.0091)} = 60 * 0.761 = 45.6^\circ C \Rightarrow T = 65.6^\circ C$$

$$b) \quad T - T_s = 60 e^{120(-0.0091)} = 60 * 0.335 = 20.1^\circ C \Rightarrow T = 40.1^\circ C$$

$$c) \quad 10 = 60 e^{-0.0091 t} \Rightarrow -0.0091 t = -\ln 6 \Rightarrow t = 3.3 \text{ hrs.}$$

2-2- Trigonometric functions : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$\sin \theta = \frac{y}{r} = \frac{1}{\csc \theta}, \quad \cos \theta = \frac{x}{r} = \frac{1}{\sec \theta}, \quad \tan \theta = \frac{y}{x} = \frac{1}{\cot \theta} = \frac{\sin \theta}{\cos \theta}$$



The following are some properties of these functions :

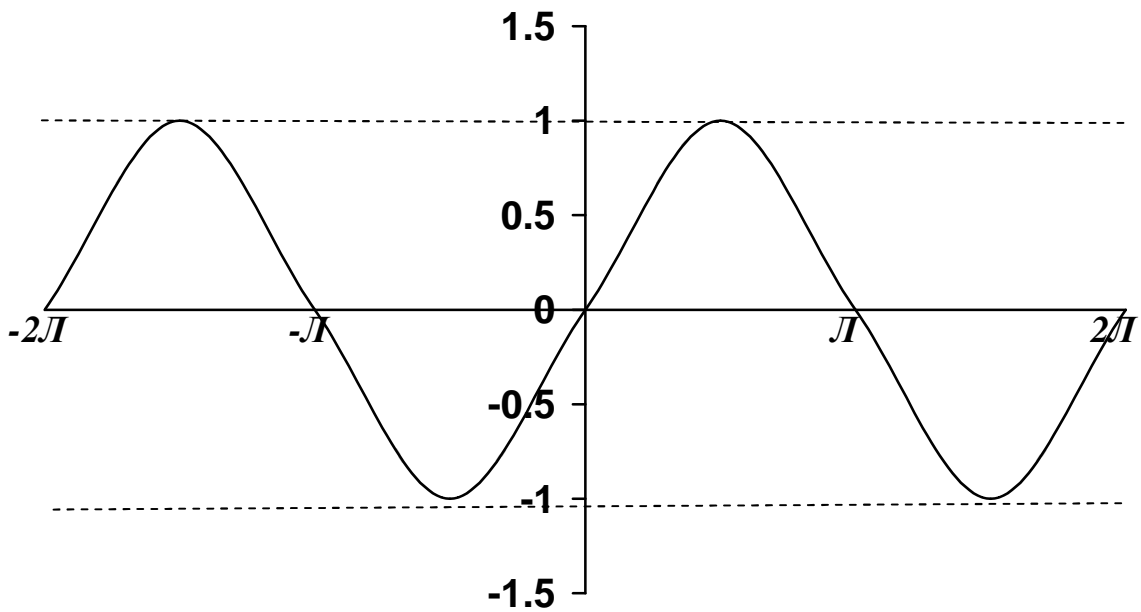
- 1) $\sin^2 \theta + \cos^2 \theta = 1$
- 2) $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$
- 3) $\sin(\theta \mp \beta) = \sin \theta \cdot \cos \beta \mp \cos \theta \cdot \sin \beta$
- 4) $\cos(\theta \mp \beta) = \cos \theta \cdot \cos \beta \pm \sin \theta \cdot \sin \beta$
- 5) $\tan(\theta \mp \beta) = \frac{\tan \theta \mp \tan \beta}{1 \pm \tan \theta \cdot \tan \beta}$
- 6) $\sin 2\theta = 2 \sin \theta \cdot \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
- 7) $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ and $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$
- 8) $\sin(\theta \mp \frac{\pi}{2}) = \mp \cos \theta$ and $\cos(\theta \mp \frac{\pi}{2}) = \pm \sin \theta$
- 9) $\sin(-\theta) = -\sin \theta$ and $\cos(-\theta) = \cos \theta$ and $\tan(-\theta) = -\tan \theta$
- 10) $\sin \theta \cdot \sin \beta = \frac{1}{2} [\cos(\theta - \beta) - \cos(\theta + \beta)]$
 $\cos \theta \cdot \cos \beta = \frac{1}{2} [\cos(\theta - \beta) + \cos(\theta + \beta)]$
 $\sin \theta \cdot \cos \beta = \frac{1}{2} [\sin(\theta - \beta) + \sin(\theta + \beta)]$

$$11) \quad \begin{aligned} \sin \theta + \sin \beta &= 2 \sin \frac{\theta + \beta}{2} \cdot \cos \frac{\theta - \beta}{2} \\ \sin \theta - \sin \beta &= 2 \cos \frac{\theta + \beta}{2} \cdot \sin \frac{\theta - \beta}{2} \end{aligned}$$

$$12) \quad \begin{aligned} \cos \theta + \cos \beta &= 2 \cos \frac{\theta + \beta}{2} \cdot \cos \frac{\theta - \beta}{2} \\ \cos \theta - \cos \beta &= -2 \sin \frac{\theta + \beta}{2} \cdot \sin \frac{\theta - \beta}{2} \end{aligned}$$

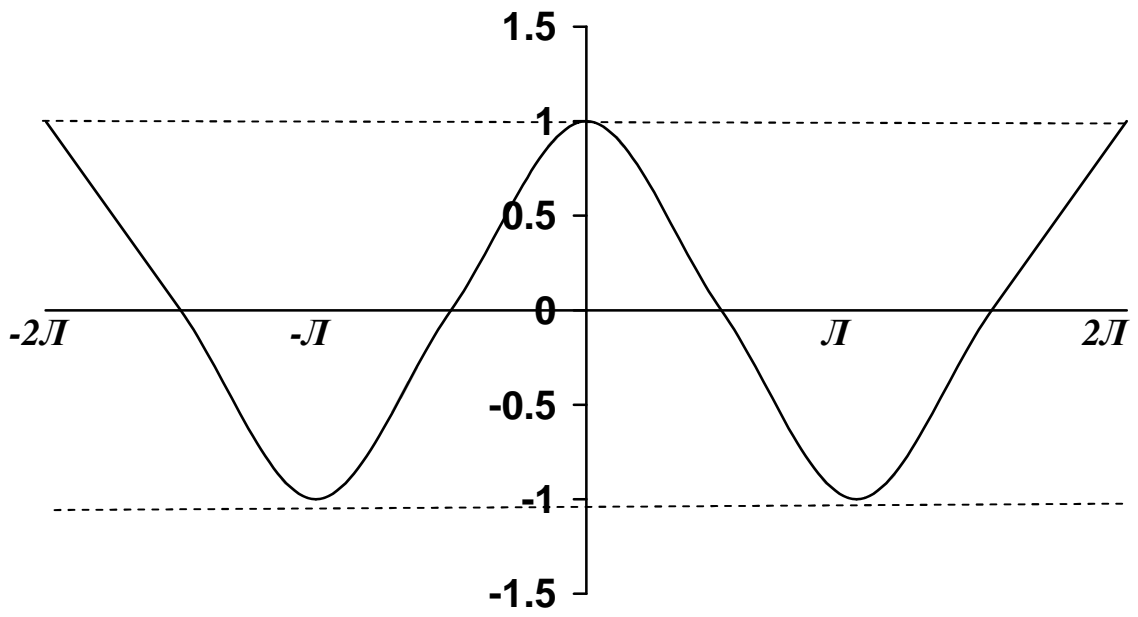
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin \theta$	0	1/2	1/√2	√3/2	1	0
$\cos \theta$	1	√3/2	1/√2	1/2	0	-1
$\tan \theta$	0	1/√2	1	√3	∞	0

Graphs of the trigonometric functions are :



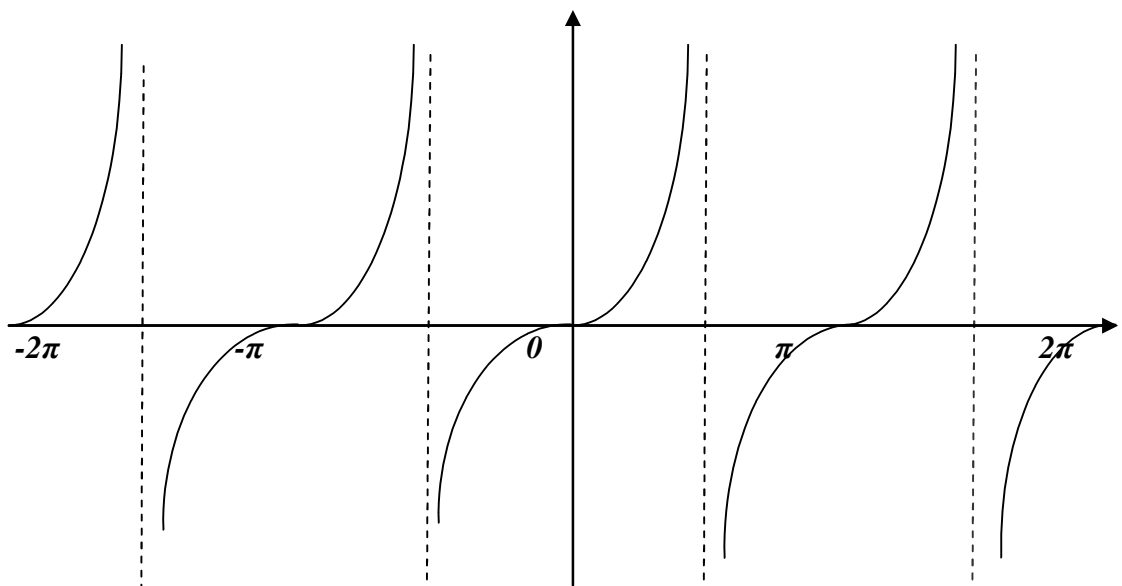
$$y = \sin x \quad D_x : \forall x$$

$$R_y : -1 \leq y \leq 1$$



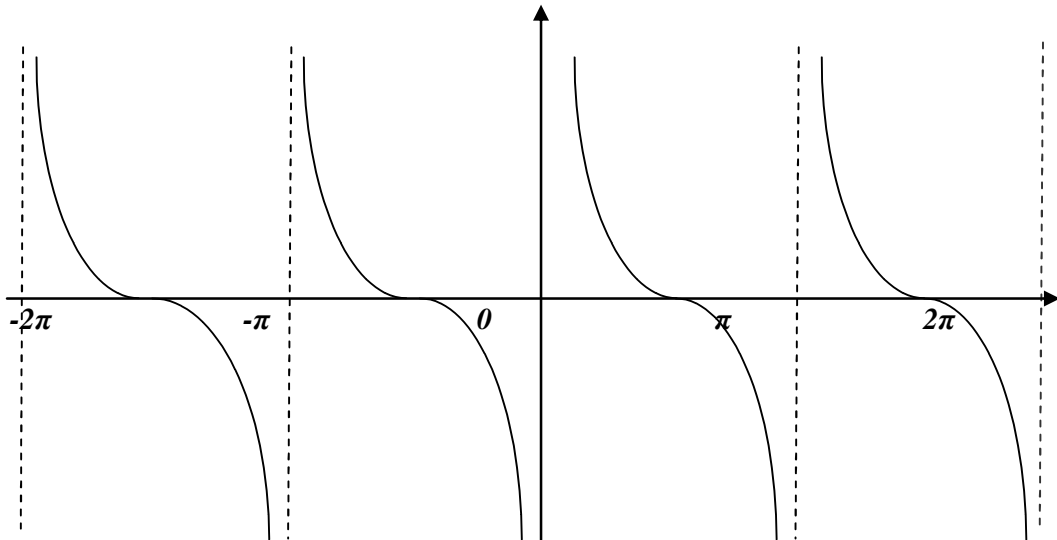
$$y = \cos x \quad D_x : \forall x$$

$$R_y : -1 \leq y \leq 1$$



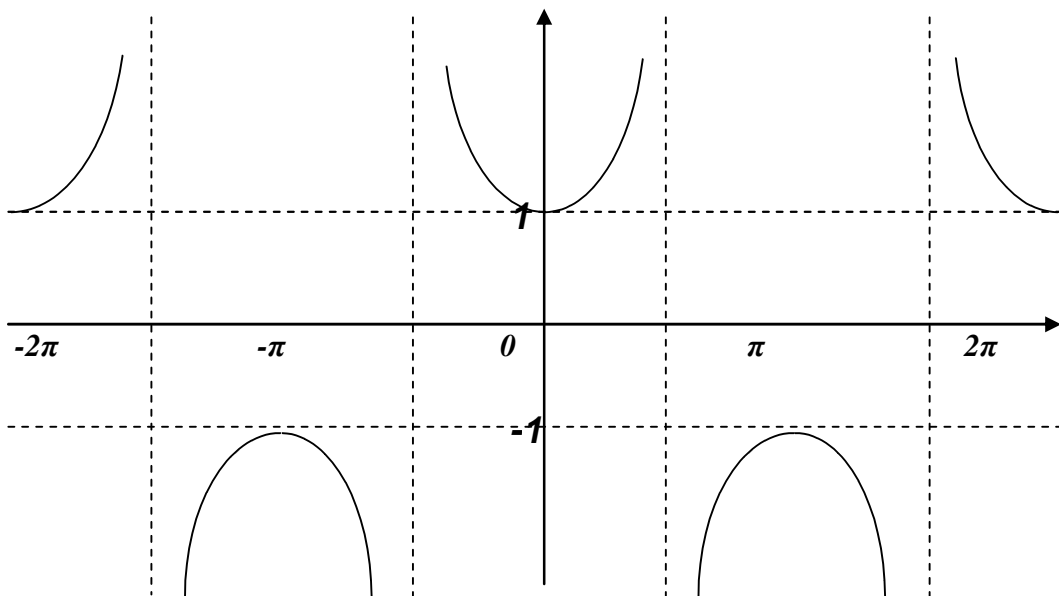
$$y = \tan x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$

$$R_y : \forall y$$



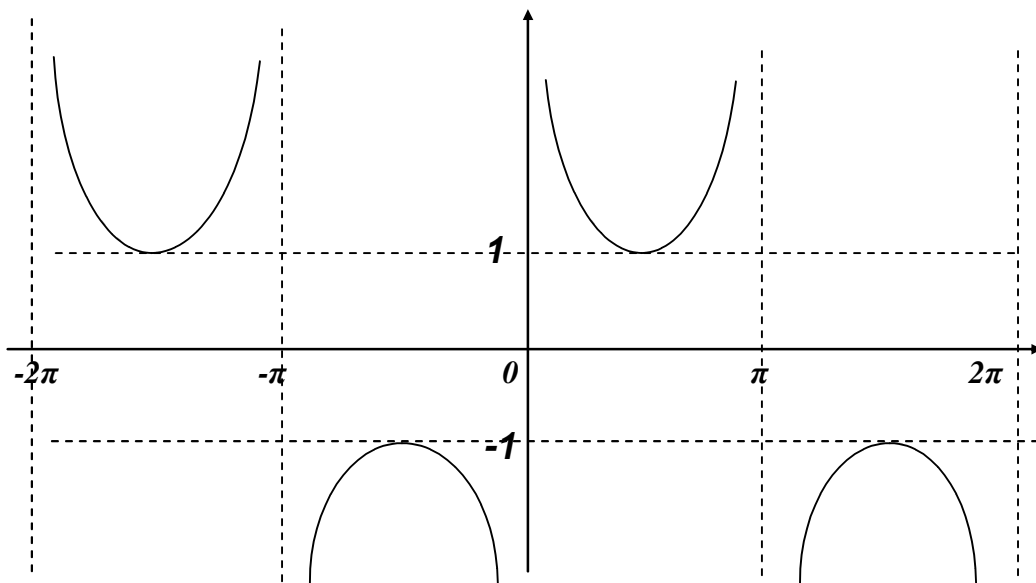
$$y = \text{Cot}x \quad D_x : \forall x \neq n\pi$$

$$R_y : \forall y$$



$$y = \text{Sec}x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$

$$R_y : \forall y \geq 1 \text{ or } y \leq -1$$



$$y = \csc x \quad D_x : \forall x \neq n\pi$$

$$R_y : \forall y \geq 1 \text{ or } y \leq -1$$

Where $n = 0, \pm 1, \pm 2, \pm 3, \dots$

EX-2 - Solve the following equations , for values of θ from 0° to 360° inclusive .

a) $\tan \theta = 2 \sin \theta$ b) $1 + \cos \theta = 2 \sin^2 \theta$

Sol.-

$$a) \quad \tan \theta = 2 \sin \theta \Rightarrow \frac{\sin \theta}{\cos \theta} = 2 \sin \theta$$

$$\Rightarrow \sin \theta (1 - 2 \cos \theta) = 0$$

$$\text{either } \sin \theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ, 360^\circ$$

$$\text{or } \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ, 300^\circ$$

Therefore the required values of θ are $0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$.

$$b) \quad 1 + \cos \theta = 2 \sin^2 \theta \Rightarrow 1 + \cos \theta = 2(1 - \cos^2 \theta)$$

$$\Rightarrow (2 \cos \theta - 1)(\cos \theta + 1) = 0$$

$$\text{either } \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ, 300^\circ$$

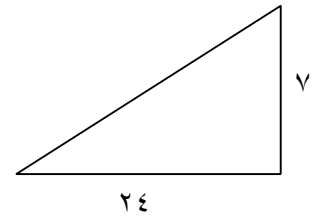
$$\text{or } \cos \theta = -1 \Rightarrow \theta = 180^\circ$$

There the roots of the equation between 0° and 360° are $60^\circ, 180^\circ$ and 300° .

EX-3- If $\tan \theta = 7/24$, find without using tables the values of $\sec \theta$ and $\sin \theta$.
Sol.-

$$\tan \theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$\sec \theta = \frac{r}{x} = \frac{25}{24} \quad \text{and} \quad \sin \theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- Prove the following identities :

- a) $\csc \theta + \tan \theta \cdot \sec \theta = \csc \theta \cdot \sec^2 \theta$
 b) $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$
 c) $\frac{\sec \theta - \csc \theta}{\tan \theta - \cot \theta} = \frac{\tan \theta + \cot \theta}{\sec \theta + \csc \theta}$

Sol.-

a) $L.H.S. = \csc \theta + \tan \theta \cdot \sec \theta = \frac{1}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta}$
 $= \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cdot \cos^2 \theta} = \frac{1}{\sin \theta} \cdot \frac{1}{\cos^2 \theta} = \csc \theta \cdot \sec^2 \theta = R.H.S.$

b) $L.H.S. = \cos^4 \theta - \sin^4 \theta = (\cos^2 \theta - \sin^2 \theta) \cdot (\cos^2 \theta + \sin^2 \theta)$
 $= \cos^2 \theta - \sin^2 \theta = R.H.S.$

c) $L.H.S. = \frac{\sec \theta - \csc \theta}{\tan \theta - \cot \theta} = \frac{\frac{1}{\cos \theta} - \frac{1}{\sin \theta}}{\frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta}} = \frac{1}{\sin \theta + \cos \theta}$
 $= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta + \cos \theta} \cdot \frac{\frac{1}{\sin \theta \cdot \cos \theta}}{1} = \frac{\tan \theta + \cot \theta}{\sec \theta + \csc \theta} = R.H.S.$

EX-5- Simplify $\frac{1}{\sqrt{x^2 - a^2}}$ when $x = a \cdot \csc \theta$.

Sol.- $\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2 \csc^2 \theta - a^2}} = \frac{1}{a \sqrt{\cot^2 \theta}} = \frac{1}{a} \tan \theta$.

EX-6- Eliminate θ from the equations :

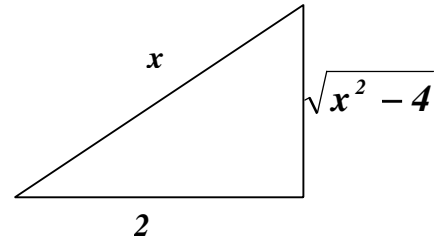
- i) $x = a \sin \theta$ and $y = b \tan \theta$
 ii) $x = 2 \sec \theta$ and $y = \cos 2\theta$

Sol.-

$$i) \quad x = a \cdot \sin \theta \Rightarrow \sin \theta = \frac{x}{a} \Rightarrow \csc \theta = \frac{a}{x}$$

$$y = b \tan \theta \Rightarrow \tan \theta = \frac{y}{b} \Rightarrow \cot \theta = \frac{b}{y}$$

$$\text{Since } \csc^2 \theta = \cot^2 \theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$



$$ii) \quad x = 2 \sec \theta \Rightarrow \cos \theta = \frac{2}{x}$$

$$y = \cos 2\theta \Rightarrow y = \cos^2 \theta - \sin^2 \theta$$

$$y = \frac{4}{x^2} - \frac{x^2 - 4}{x^2} \Rightarrow x^2 y = 8 - x^2$$

EX-7- If $\tan^2 \theta - 2 \tan^2 \beta = 1$, show that $2 \cos^2 \theta - \cos^2 \beta = 0$.

Sol. -

$$\tan^2 \theta - 2 \tan^2 \beta = 1 \Rightarrow \sec^2 \theta - 1 - 2(\sec^2 \beta - 1) = 1$$

$$\Rightarrow \sec^2 \theta - 2 \sec^2 \beta = 0 \Rightarrow \frac{1}{\cos^2 \theta} - \frac{2}{\cos^2 \beta} = 0$$

$$\Rightarrow 2 \cos^2 \theta - \cos^2 \beta = 0 \quad \text{Q.E.D.}$$

EX-8- If $a \sin \theta = p - b \cos \theta$ and $b \sin \theta = q + a \cos \theta$. Show that :
 $a^2 + b^2 = p^2 + q^2$

Sol. -

$$p = a \cdot \sin \theta + b \cdot \cos \theta \quad \text{and} \quad q = b \cdot \sin \theta - a \cdot \cos \theta$$

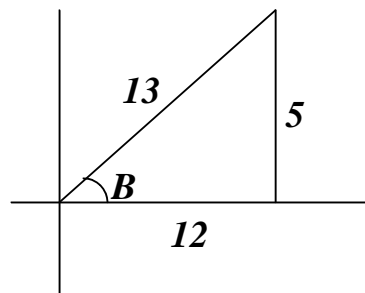
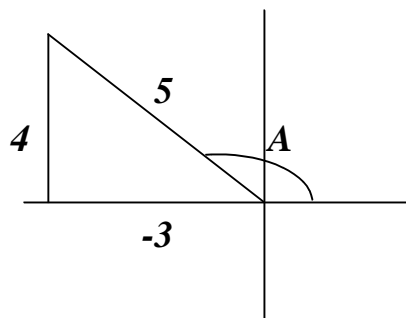
$$p^2 + q^2 = (a \sin \theta + b \cos \theta)^2 + (b \sin \theta - a \cos \theta)^2$$

$$= a^2 (\sin^2 \theta + \cos^2 \theta) + b^2 (\cos^2 \theta + \sin^2 \theta) = a^2 + b^2$$

EX-9- If $\sin A = 4/5$ and $\cos B = 12/13$, where A is obtuse and B is acute. Find, without tables, the values of :

a) $\sin(A - B)$, b) $\tan(A - B)$, c) $\tan(A + B)$.

Sol. -



$$a) \quad \sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65}$$

$$b) \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$= \frac{\frac{4}{3} - \frac{5}{12}}{1 + \frac{4}{3} \cdot \frac{5}{12}} = -\frac{63}{16}$$

$$c) \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$= \frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3} \cdot \frac{5}{12}} = \frac{33}{56}$$

EX-10 – Prove the following identities:

$$a) \quad \sin(A + B) + \sin(A - B) = 2 \cdot \sin A \cdot \cos B$$

$$b) \quad \tan A + \tan B = \frac{\sin(A + B)}{\cos A \cdot \cos B}$$

$$c) \quad \sec(A + B) = \frac{\sec A \cdot \sec B \cdot \csc A \cdot \csc B}{\csc A \cdot \csc B - \sec A \cdot \sec B}$$

$$d) \quad \frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \cot \theta$$

Sol.-

$$\begin{aligned} a) \quad L.H.S. &= \sin(A+B) + \sin(A-B) \\ &= \sin A \cdot \cos B + \cos A \cdot \sin B + \sin A \cdot \cos B - \cos A \cdot \sin B \\ &= 2 \cdot \sin A \cdot \cos B = R.H.S. \end{aligned}$$

$$\begin{aligned} b) \quad R.H.S. &= \frac{\sin(A+B)}{\cos A \cdot \cos B} = \frac{\sin A \cdot \cos B + \cos A \cdot \sin B}{\cos A \cdot \cos B} \\ &= \tan A + \tan B = L.H.S. \end{aligned}$$

$$\begin{aligned} c) \quad R.H.S. &= \frac{\sec A \cdot \sec B \cdot \csc A \cdot \csc B}{\csc A \cdot \csc B - \sec A \cdot \sec B} = \frac{\frac{1}{\cos A} \cdot \frac{1}{\cos B} \cdot \frac{1}{\sin A} \cdot \frac{1}{\sin B}}{\frac{1}{\sin A} \cdot \frac{1}{\sin B} - \frac{1}{\cos A} \cdot \frac{1}{\cos B}} \\ &= \frac{1}{\cos A \cdot \cos B - \sin A \cdot \sin B} = \frac{1}{\cos(A+B)} \\ &= \sec(A+B) = L.H.S. \end{aligned}$$

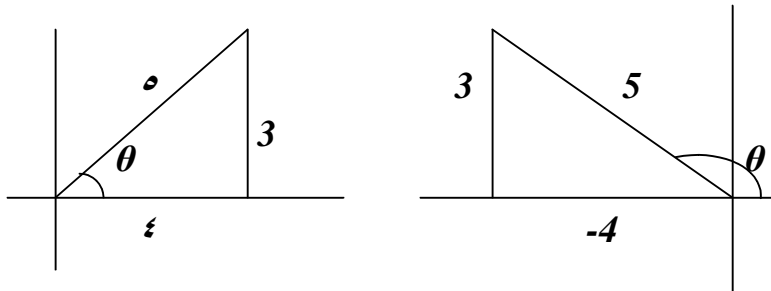
$$\begin{aligned} d) \quad L.H.S. &= \frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \frac{2 \sin \theta \cdot \cos \theta + (\cos^2 \theta - \sin^2 \theta) + 1}{2 \sin \theta \cdot \cos \theta - (\cos^2 \theta - \sin^2 \theta) + 1} \\ &= \frac{2 \sin \theta \cdot \cos \theta + 2 \cos^2 \theta}{2 \sin \theta \cdot \cos \theta + 2 \sin^2 \theta} = \frac{\cos \theta}{\sin \theta} = \cot \theta = R.H.S. \end{aligned}$$

EX-11 – Find, without using tables, the values of $\sin 2\theta$ and $\cos 2\theta$, when:

a) $\sin \theta = 3/5$, b) $\cos \theta = 12/13$, c) $\sin \theta = -\sqrt{3}/2$.

Sol. –

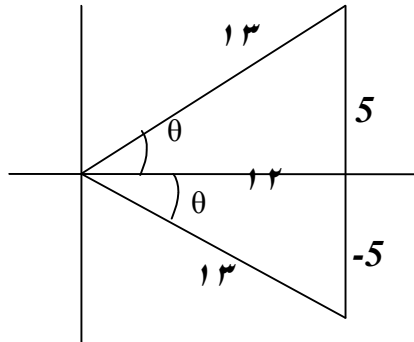
a)



$$\sin 2\theta = 2 \cdot \sin \theta \cdot \cos \theta = 2 \cdot \frac{3}{5} \cdot \left(\pm \frac{4}{5}\right) = \pm \frac{24}{25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\pm \frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

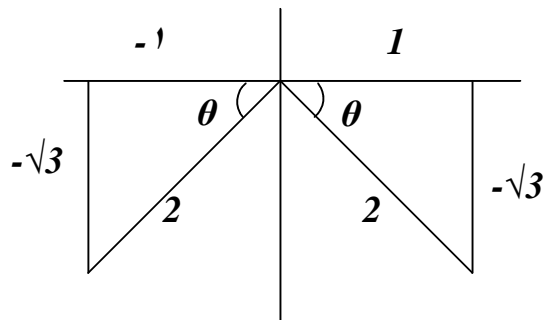
b)



$$\sin 2\theta = 2 \sin \theta \cos \theta = 2\left(\pm \frac{5}{13}\right) \cdot \left(\frac{12}{13}\right) = \pm \frac{120}{169}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\frac{12}{13}\right)^2 - \left(\pm \frac{5}{13}\right)^2 = \frac{119}{169}$$

c)



$$\sin 2\theta = 2 \sin \theta \cos \theta = 2\left(-\frac{\sqrt{3}}{2}\right) \cdot \left(\mp \frac{1}{2}\right) = \pm \frac{\sqrt{3}}{2}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\mp \frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2}$$

EX-12- Solve the following equations for values of θ from 0° to 360° inclusive:

a) $\cos 2\theta + \cos \theta + 1 = 0$, b) $4 \tan \theta \cdot \tan 2\theta = 1$

Sol.-

$$a) \quad \cos 2\theta + \cos \theta + 1 = 0 \Rightarrow 2\cos^2 \theta - 1 + \cos \theta + 1 = 0$$

$$\Rightarrow \cos(2\cos \theta + 1) = 0$$

$$\text{either } \cos \theta = 0 \Rightarrow \theta = 90^\circ, 270^\circ$$

$$\text{or } \cos \theta = -\frac{1}{2} \Rightarrow \theta = 120^\circ, 240^\circ$$

$$\theta = \{90^\circ, 120^\circ, 240^\circ, 270^\circ\}$$

$$b) \quad 4 \cdot \tan \theta \cdot \tan 2\theta = 1 \Rightarrow 4 \cdot \tan \theta \cdot \frac{2 \tan \theta}{1 - \tan^2 \theta} = 1$$

$$\Rightarrow 9 \tan^2 \theta = 1$$

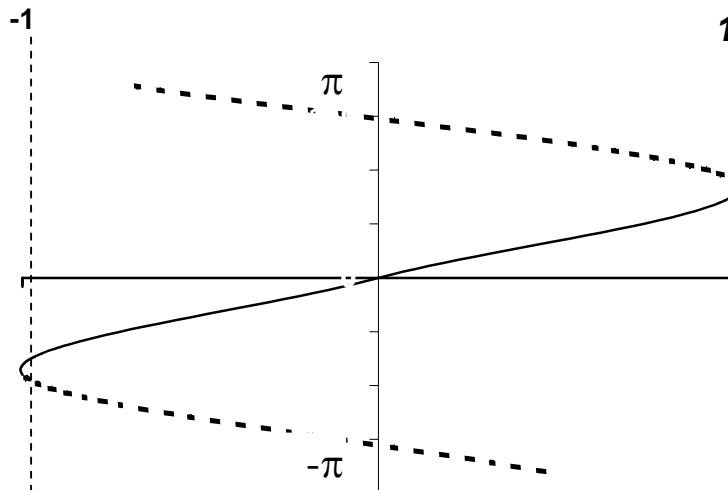
$$\text{either } \tan \theta = \frac{1}{3} \Rightarrow \theta = 18.4^\circ, 198.4^\circ$$

$$\text{or } \tan \theta = -\frac{1}{3} \Rightarrow \theta = 161.6^\circ, 341.6^\circ$$

$$\theta = \{18.4^\circ, 161.6^\circ, 198.4^\circ, 341.6^\circ\}$$

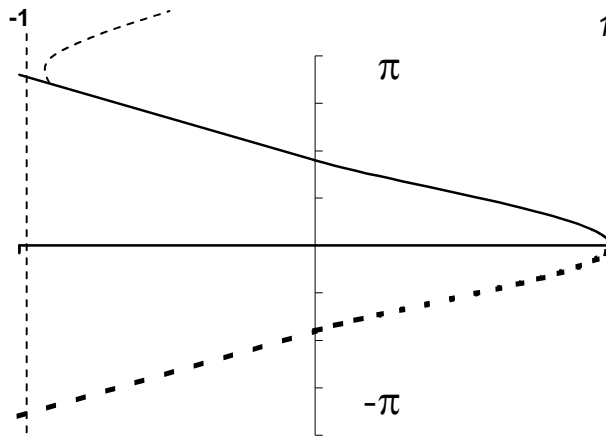
2-3- The inverse trigonometric functions : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :

$$y = \sin x \Leftrightarrow x = \sin^{-1} y$$



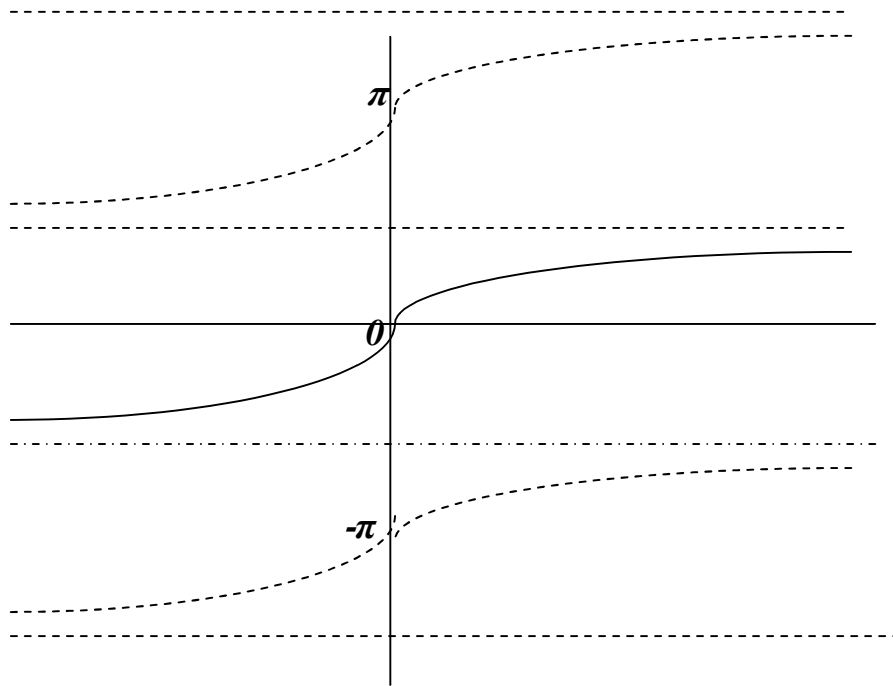
$$y = \sin^{-1} x \quad D_x : -1 \leq x \leq 1$$

$$R_y : -90 \leq y \leq 90$$



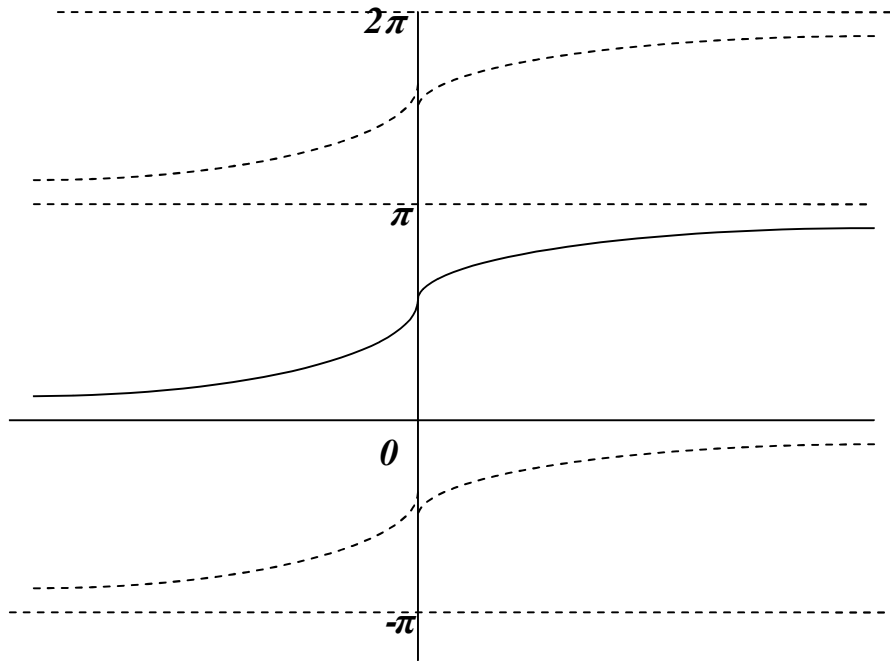
$$y = \text{Cos}^{-1} x \quad D_x : -1 \leq x \leq 1$$

$$R_y : 0 \leq y \leq 180$$



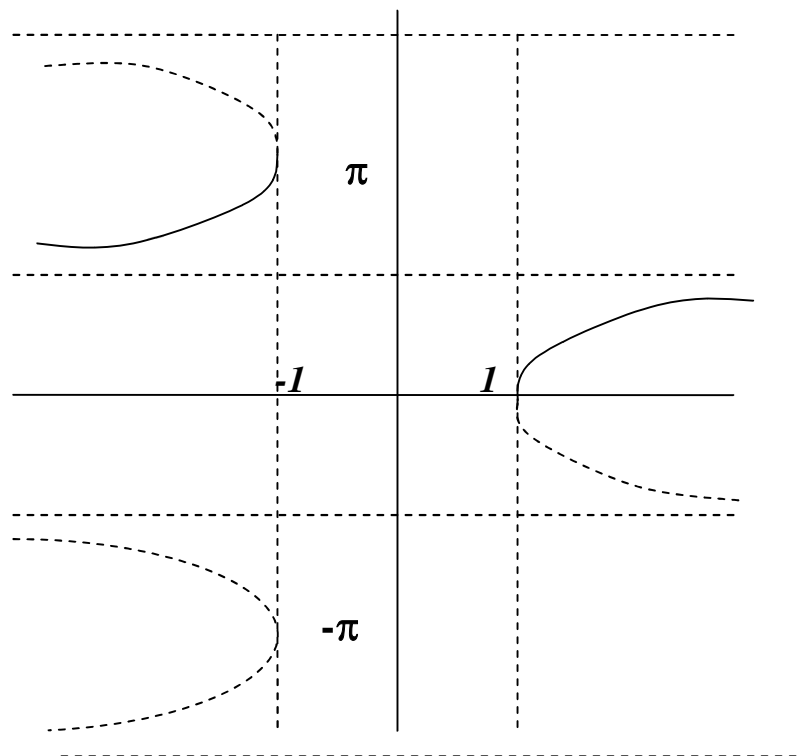
$$y = \text{tan}^{-1} x \quad D_x : \forall x$$

$$R_y : -90 \leq y \leq 90$$



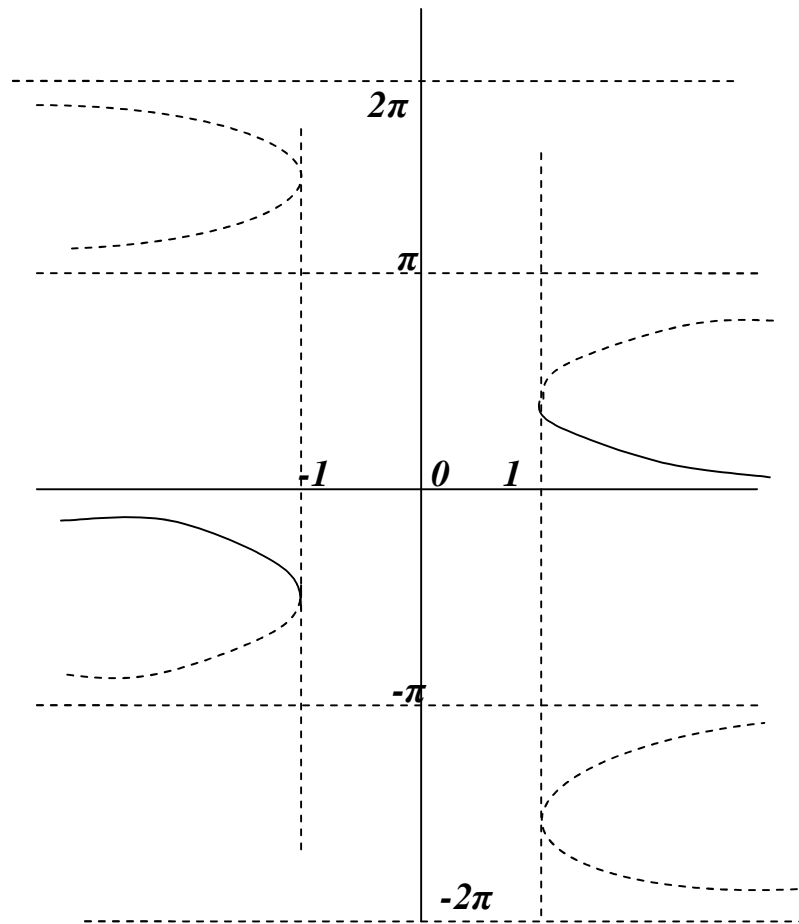
$$y = \text{Cot}^{-1} x \quad D_x : \forall x$$

$$R_y : 0 \leq y \leq \pi$$



$$y = \text{Sec}^{-1} x \quad D_x : \forall |x| \geq 1$$

$$R_y : 0 \leq y \leq \pi, y \neq \frac{\pi}{2}$$



$$y = \text{Csc}^{-1} x \quad D_x : \forall |x| \geq 1$$

$$R_y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$$

The following are some properties of the inverse trigonometric functions :

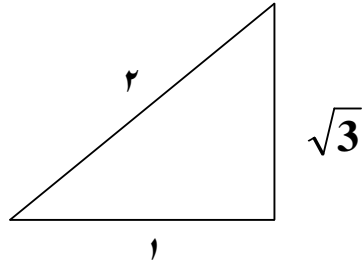
1. $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$
2. $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x$
3. $\text{Sin}^{-1} x + \text{Cos}^{-1} x = \frac{\pi}{2}$
4. $\text{tan}^{-1}(-x) = -\text{tan}^{-1} x$
5. $\text{Cot}^{-1} x = \frac{\pi}{2} - \text{tan}^{-1} x$
6. $\text{Sec}^{-1} x = \text{Cos}^{-1} \frac{1}{x}$
7. $\text{Csc}^{-1} x = \text{Sin}^{-1} \frac{1}{x}$
8. $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1} x$

and noted that $(\text{Sin} x)^{-1} = \frac{1}{\text{Sin} x} = \text{Csc} x \neq \text{Sin}^{-1} x$

EX-13- Given that $\alpha = \text{Sin}^{-1} \frac{\sqrt{3}}{2}$, find :

$\text{Csc} \alpha$, $\text{Cos} \alpha$, $\text{Sec} \alpha$, $\text{tan} \alpha$, and $\text{Cot} \alpha$

Sol.-



$$\alpha = \text{Sin}^{-1} \frac{\sqrt{3}}{2} \Rightarrow \text{Sin} \alpha = \frac{\sqrt{3}}{2} = \frac{x}{y} \Rightarrow r = \sqrt{4-3} = 1$$

$$\text{Csc} \alpha = \frac{2}{\sqrt{3}}, \text{Cos} \alpha = \frac{1}{2}, \text{Sec} \alpha = 2, \text{tan} \alpha = \sqrt{3}, \text{Cot} \alpha = \frac{1}{\sqrt{3}}$$

EX-14 – Evaluate the following expressions :

a) $\text{Sec}(\text{Cos}^{-1} \frac{1}{2})$ b) $\text{Sin}^{-1} 1 - \text{Sin}^{-1}(-1)$ c) $\text{Cos}^{-1}(-\text{Sin} \frac{\pi}{6})$

Sol.-

a) $\text{Sec}(\text{Cos}^{-1} \frac{1}{2}) = \text{Sec} \frac{\pi}{3} = 2$

b) $\text{Sin}^{-1} 1 - \text{Sin}^{-1}(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$

c) $\text{Cos}^{-1}(-\text{Sin} \frac{\pi}{6}) = \text{Cos}^{-1}(-\frac{1}{2}) = \frac{2}{3} \pi$

EX-15- Prove that :

a) $\text{Sec}^{-1} x = \text{Cos}^{-1} \frac{1}{x}$ b) $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$

Sol.

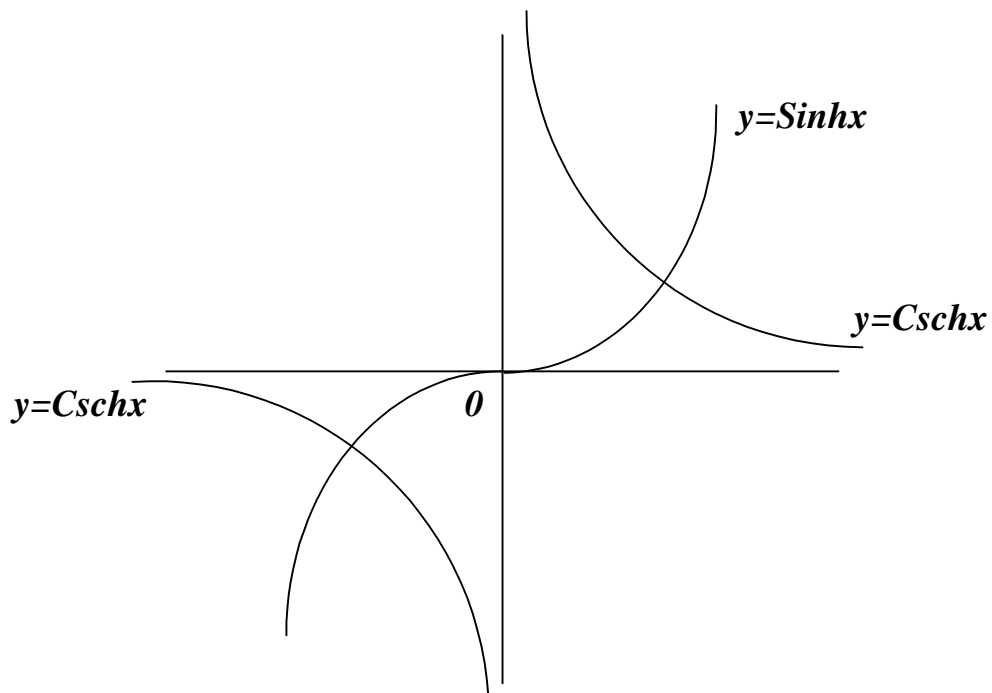
a) Let $y = \text{Sec}^{-1} x \Rightarrow x = \text{Sec} y \Rightarrow x = \frac{1}{\text{Cos} y}$
 $\Rightarrow y = \text{Cos}^{-1} \frac{1}{x} \Rightarrow \text{Sec}^{-1} x = \text{Cos}^{-1} \frac{1}{x}$

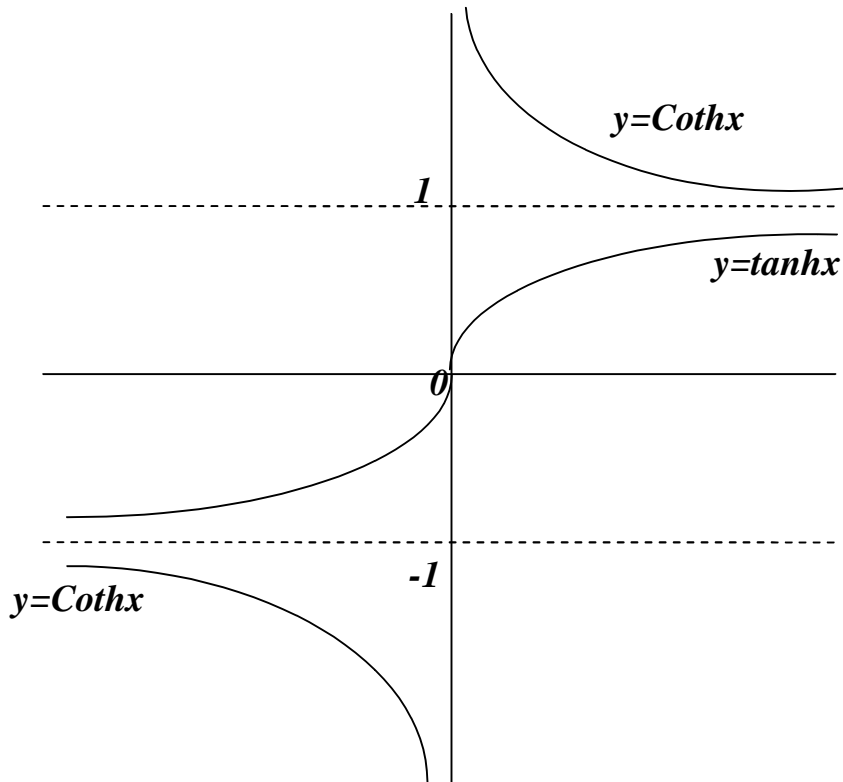
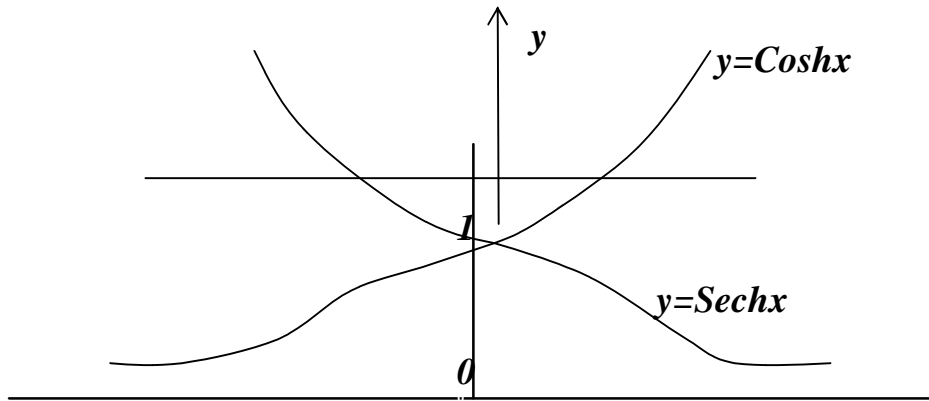
b) Let $y = -\text{Sin}^{-1} x \Rightarrow x = \text{Sin}(-y) \Rightarrow x = -\text{Sin} y$
 $\Rightarrow y = \text{Sin}^{-1}(-x) \Rightarrow \text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$

2-4- Hyperbolic functions : Hyperbolic functions are used to describe the motions of waves in elastic solids ; the shapes of electric power lines ; temperature distributions in metal fins that cool pipes ...etc.

The hyperbolic sine (Sinh) and hyperbolic cosine (Cosh) are defined by the following equations :

1. $\text{Sinhu} = \frac{1}{2}(e^u - e^{-u})$ and $\text{Coshu} = \frac{1}{2}(e^u + e^{-u})$
2. $\tanh u = \frac{\text{Sinhu}}{\text{Coshu}} = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ and $\text{Cothu} = \frac{\text{Coshu}}{\text{Sinhu}} = \frac{e^u + e^{-u}}{e^u - e^{-u}}$
3. $\text{Sechu} = \frac{1}{\text{Coshu}} = \frac{2}{e^u + e^{-u}}$ and $\text{Cschu} = \frac{1}{\text{Sinhu}} = \frac{2}{e^u - e^{-u}}$
4. $\text{Cosh}^2 u - \text{Sinh}^2 u = 1$
5. $\tanh^2 u + \text{Sech}^2 u = 1$ and $\text{Coth}^2 u - \text{Csch}^2 u = 1$
6. $\text{Coshu} + \text{Sinhu} = e^u$ and $\text{Coshu} - \text{Sinhu} = e^{-u}$
7. $\text{Cosh}(-u) = \text{Coshu}$ and $\text{Sinh}(-u) = -\text{Sinhu}$
8. $\text{Cosh}0 = 1$ and $\text{Sinh}0 = 0$
9. $\text{Sinh}(x + y) = \text{Sinh}x.\text{Cosh}y + \text{Cosh}x.\text{Sinhy}$
10. $\text{Cosh}(x + y) = \text{Cosh}x.\text{Cosh}y + \text{Sinh}x.\text{Sinhy}$
11. $\text{Sinh}2x = 2.\text{Sinh}x.\text{Cosh}x$
12. $\text{Cosh}2x = \text{Cosh}^2 x + \text{Sinh}^2 x$
13. $\text{Cosh}^2 x = \frac{\text{Cosh}2x + 1}{2}$ and $\text{Sinh}^2 x = \frac{\text{Cosh}2x - 1}{2}$





$y = \text{Sinh } x$	$D_x : \forall x$	and	$R_y : \forall y$
$y = \text{Cosh } x$	$D_x : \forall x$	and	$R_y : y \geq 1$
$y = \text{tanh } x$	$D_x : \forall x$	and	$R_y : -1 \leq y \leq 1$
$y = \text{Coth } x$	$D_x : \forall x \neq 0$	and	$R_y : y < -1 \text{ or } y > 1$
$y = \text{Sech } x$	$D_x : \forall x$	and	$R_y : 0 < y \leq 1$
$y = \text{Csch } x$	$D_x : \forall x \neq 0$	and	$R_y : \forall y \neq 0$

EX-16- Let $\tanh u = -7/25$, determine the values of the remaining five hyperbolic functions.

Sol.-

$$\text{Cothu} = \frac{1}{\tanh u} = -\frac{25}{7}$$

$$\tanh^2 u + \text{Sech}^2 u = 1 \Rightarrow \frac{49}{625} + \text{Sech}^2 u = 1 \Rightarrow \text{Sechu} = \frac{24}{25}$$

$$\text{Coshu} = \frac{1}{\text{Sechu}} = \frac{25}{24}$$

$$\tanh u = \frac{\text{Sinhu}}{\text{Coshu}} \Rightarrow -\frac{7}{25} = \frac{\text{Sinhu}}{\frac{25}{24}} \Rightarrow \text{Sinhu} = -\frac{7}{24}$$

$$\text{Cschu} = \frac{1}{\text{Sinhu}} = -\frac{24}{7}$$

EX-17- Rewrite the following expressions in terms of exponentials .

Write the final result as simply as you can :

a) $2\text{Cosh}(\ln x)$

b) $\tanh(\ln x)$

c) $\text{Cosh}5x + \text{Sinh}5x$

d) $(\text{Sinh}x + \text{Cosh}x)^4$

Sol.-

$$a) \quad 2\text{Cosh}(\ln x) = 2 \cdot \frac{e^{\ln x} + e^{-\ln x}}{2} = x + \frac{1}{x}$$

$$b) \quad \tanh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{x^2 - 1}{x^2 + 1}$$

$$c) \quad \text{Cosh}5x + \text{Sinh}5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$$

$$d) \quad (\text{Sinh}x + \text{Cosh}x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \right)^4 = e^{4x}$$

EX-18- Solve the equation for x : $\text{Cosh} x = \text{Sinh} x + 1/2$.

Sol. - $\text{Cosh}x - \text{Sinh}x = \frac{1}{2} \Rightarrow e^{-x} = \frac{1}{2} \Rightarrow -x = \ln 1 - \ln 2 \Rightarrow x = \ln 2$

EX-19 – Verify the following identity :

a) $\text{Sinh}(u+v) = \text{Sinh} u \cdot \text{Cosh} v + \text{Cosh} u \cdot \text{Sinh} v$

b) then verify $\text{Sinh}(u-v) = \text{Sinh} u \cdot \text{Cosh} v - \text{Cosh} u \cdot \text{Sinh} v$

Sol.-

$$\begin{aligned}
 a) \text{ R.H.S.} &= \text{Sinhu.Coshv} + \text{Coshu.Sinhv} \\
 &= \frac{e^u - e^{-u}}{2} \cdot \frac{e^v + e^{-v}}{2} + \frac{e^u + e^{-u}}{2} \cdot \frac{e^v - e^{-v}}{2} \\
 &= \frac{e^{u+v} - e^{-(u+v)}}{2} = \text{Sinh}(u+v) = \text{L.H.S.}
 \end{aligned}$$

$$\begin{aligned}
 b) \text{ L.H.S.} &= \text{Sinh}(u + (-v)) = \text{Sinhu.Cosh}(-v) + \text{Coshu.Sinh}(-v) \\
 &= \text{Sinhu.Coshv} - \text{Coshu.Sinhv} = \text{R.H.S.}
 \end{aligned}$$

EX-20 – Verify the following identities :

$$a) \quad \text{Sinhu.Coshv} = \frac{1}{2} [\text{Sinh}(u+v) + \text{Sinh}(u-v)]$$

$$b) \quad \text{Coshu.Coshv} = \frac{1}{2} [\text{Cosh}(u+v) + \text{Cosh}(u-v)]$$

$$c) \quad \text{Sinh}3u = \text{Sinh}^3u + 3\text{Cosh}^2u.\text{Sinhu} = 3\text{Sinhu} + 4\text{Sinh}^3u$$

$$d) \quad \text{Sinh}^2u - \text{Sinh}^2v = \text{Cosh}^2u - \text{Cosh}^2v$$

Sol. –

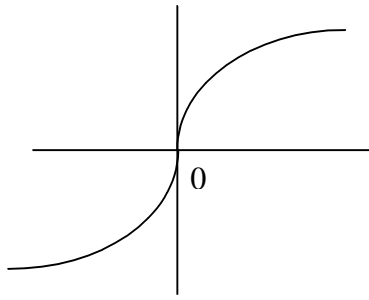
$$\begin{aligned}
 a) \text{ R.H.S.} &= \frac{1}{2} [\text{Sinh}(u+v) + \text{Sinh}(u-v)] \\
 &= \frac{1}{2} [\text{Sinhu.Coshv} + \text{Coshu.Sinhv} + \text{Sinhu.Coshv} - \text{Coshu.Sinhv}] \\
 &= \text{Sinhu.Coshv} = \text{L.H.S.}
 \end{aligned}$$

$$\begin{aligned}
 b) \text{ R.H.S.} &= \frac{1}{2} [\text{Cosh}(u+v) + \text{Cosh}(u-v)] \\
 &= \frac{1}{2} [\text{Coshu.Coshv} + \text{Sinhu.Sinhv} + \text{Coshu.Coshv} - \text{Sinhu.Sinhv}] \\
 &= \text{Coshu.Coshv} = \text{L.H.S.}
 \end{aligned}$$

$$\begin{aligned}
 c) \text{ L.H.S.} &= \text{Sinh}(2u+u) = \text{Sinh}2u.\text{Coshu} + \text{Cosh}2u.\text{Sinhu} \\
 &= 2\text{Sinhu.Coshu.Coshu} + (\text{Cosh}^2u + \text{Sinh}^2u).\text{Sinhu} \\
 &= 3\text{Sinhu.Cosh}^2u + \text{Sinh}^3u = \text{R.H.S.}(I) \\
 &= 3\text{Sinhu}.(1 + \text{Sinh}^2u) + \text{Sinh}^3u = 3\text{Sinhu} + 4\text{Sinh}^3u = \text{R.H.S.}(II)
 \end{aligned}$$

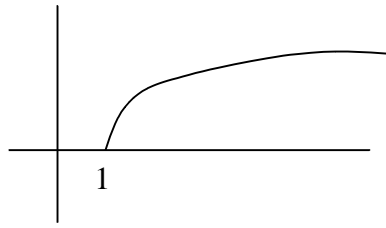
$$\begin{aligned}
 d) \text{ L.H.S.} &= \text{Sinh}^2u - \text{Sinh}^2v = \text{Cosh}^2u - 1 - (\text{Cosh}^2v - 1) \\
 &= \text{Cosh}^2u - \text{Cosh}^2v = \text{R.H.S.}
 \end{aligned}$$

2-5- Inverse hyperbolic functions : All hyperbolic functions have inverses that are useful in integration and interesting as differentiable functions in their own right .



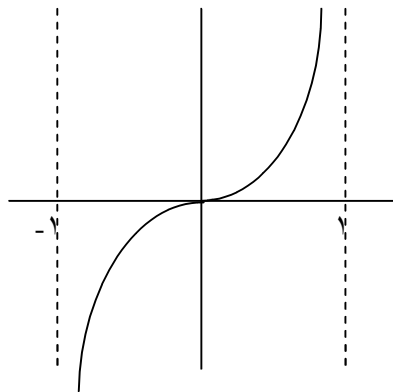
$$y = \text{Sinh}^{-1} x \quad D_x : \forall x$$

$$R_y : \forall y$$



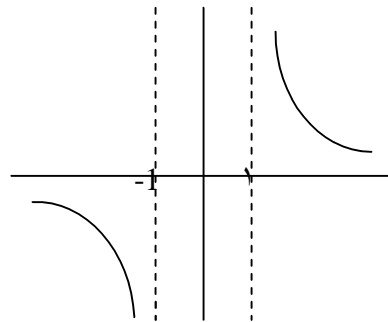
$$y = \text{Cosh}^{-1} x \quad D_x : \forall x \geq 1$$

$$R_y : \forall y \geq 0$$



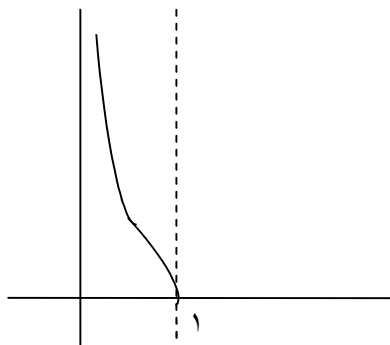
$$y = \text{tanh}^{-1} x \quad D_x : -1 < x < 1$$

$$R_y : \forall y$$



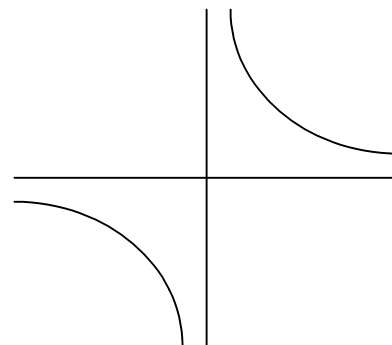
$$y = \text{Coth}^{-1} x \quad D_x : \forall x < -1 \text{ or } x > 1$$

$$R_y : \forall y \neq 0$$



$$y = \text{Sech}^{-1} x \quad D_x : 0 < x \leq 1$$

$$R_y : \forall y \geq 0$$



$$y = \text{Csch}^{-1} x \quad D_x : \forall x \neq 0$$

$$R_y : \forall y \neq 0$$

Some useful identities :

1. $\text{Sinh}^{-1} x = \ln(x + \sqrt{x^2 + 1})$
2. $\text{Cosh}^{-1} x = \ln(x + \sqrt{x^2 - 1})$
3. $\text{tanh}^{-1} x = \frac{1}{2} \cdot \ln\left(\frac{1+x}{1-x}\right)$
4. $\text{Coth}^{-1} x = \frac{1}{2} \cdot \ln\left(\frac{x+1}{x-1}\right) = \text{tanh}^{-1} \frac{1}{x}$
5. $\text{Sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) = \text{Cosh}^{-1} \frac{1}{x}$
6. $\text{Csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2 + 1}}{|x|}\right) = \text{Sinh}^{-1} \frac{1}{x}$

EX-21 - Derive the formula :

$$\text{Sinh}^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

Sol.-

$$\text{Let } y = \text{Sinh}^{-1} x \Rightarrow x = \text{Sinhy} = \frac{e^y - e^{-y}}{2} \Rightarrow x = \frac{e^{2y} - 1}{2e^y}$$

$$\Rightarrow e^{2y} - 2x \cdot e^y - 1 = 0$$

$$e^y = \frac{2x \mp \sqrt{4x^2 + 4}}{2} \Rightarrow e^y = x \mp \sqrt{x^2 + 1}$$

either $y = \ln(x - \sqrt{x^2 + 1})$ *neglected since $x - \sqrt{x^2 + 1} < 0$*

or $y = \ln(x + \sqrt{x^2 + 1})$

Problems – 2

1. A body of unknown temperature was placed in a room that was held at $30^{\circ} F$. After 10 minutes , the body's temperature was $0^{\circ} F$, and 20 minutes after the body was placed in the room the body's temperature $15^{\circ} F$. Use Newton's law of cooling to estimate the body's initial temperature .
(ans.: $-30^{\circ} F$)

2. A pan of warm water $46^{\circ} C$ was put in a refrigerator . Ten minutes later , the water's temperature was $39^{\circ} C$, 10 minutes after that , it was $33^{\circ} C$. Use Newton's law of cooling to estimate how cold the refrigerator was ?
(ans.: $-3^{\circ} C$)

3. Solve the following equations for values of θ from -180° to 180° inclusive:

<i>i) $\tan^2 \theta + \tan \theta = 0$</i>	<i>ii) $\text{Cot } \theta = 5 \text{ Cos } \theta$</i>
<i>iii) $3 \text{ Cos } \theta + 2 \text{ Sec } \theta + 7 = 0$</i>	<i>iv) $\text{Cos}^2 \theta + \text{Sin } \theta + 1 = 0$</i>

 (ans.: *i*) $-180, -45, 0, 135, 180$; *ii*) $-90, 11.5, 90, 168.5$; *iii*) $-109.5, 109.5$; *iv*) -90)

4. Solve the following equations for values of θ from 0° to 360° inclusive:

<i>i) $3 \text{ Cos } 2\theta - \text{Sin } \theta + 2 = 0$</i>	<i>ii) $3 \tan \theta = \tan 2\theta$</i>
<i>iii) $\text{Sin } 2\theta \cdot \text{Cos } \theta + \text{Sin}^2 \theta = 1$</i>	<i>iv) $3 \text{ Cot } 2\theta + \text{Cot } \theta = 1$</i>

 (ans.: *i*) $56.4, 123.6, 270$; *ii*) $0, 30, 150, 180, 210, 330, 360$; *iii*) $30, 90, 150, 270$; *iv*) $45, 121, 225, 301$)

5. If $\text{Sin } \theta = 3/5$, find without using tables the values of :

<i>i) $\text{Cos } \theta$</i>	<i>ii) $\tan \theta$</i>	<i>(ans.: i) $4/5$; ii) $3/4$)</i>
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6. Find, without using tables, the values of $\text{Cos } x$ and $\text{Sin } x$, when $\text{Cos } 2x$ is :

<i>a) $1/8$,</i>	<i>b) $7/25$,</i>	<i>c) $-119/169$</i>
------------------------------	-------------------------------	---------------------------------

 (ans.: *a*) $\mp \frac{3}{4}, \mp \frac{\sqrt{7}}{4}$; *b*) $\mp \frac{4}{5}, \mp \frac{3}{5}$; *c*) $\mp \frac{5}{13}, \mp \frac{12}{13}$)

7. If $\text{Sin } A = 3/5$ and $\text{Sin } B = 5/13$, where A and B are acute angles , find without using tables , the values of :

<i>a) $\text{Sin}(A+B)$, b) $\text{Cos}(A+B)$, c) $\text{Cot}(A+B)$</i>	<i>(ans.: $56/65$; $33/65$; $33/56$)</i>
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8. If $\tan A = -1/7$ and $\tan B = 3/4$, where A is obtuse and B is acute , find without using tables the value of $A - B$.
(ans.: 135)

9. Prove the following identities :

- i) $\sec^2 \theta + \csc^2 \theta = \sec^2 \theta \cdot \csc^2 \theta$
 ii) $\sin^2 \theta (1 + \sec^2 \theta) = \sec^2 \theta - \cos^2 \theta$
 iii) $\frac{1 + \sin \theta}{1 - \sin \theta} = (\sec \theta + \tan \theta)^2$
 iv) $\sec \theta - \sin \theta = \frac{\tan^2 \theta + \cos^2 \theta}{\sec \theta + \sin \theta}$
 v) $\frac{\cos(A - B) - \cos(A + B)}{\sin(A + B) + \sin(A - B)} = \tan B$
 vi) $\cos B - \cos A \cdot \cos(A - B) = \sin A \cdot \sin(A - B)$
 vii) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \cdot \tan B \cdot \tan C}{1 - \tan B \cdot \tan C - \tan C \cdot \tan A - \tan A \cdot \tan B}$

If A, B, C are angles of a triangle, show that :

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

viii) $\frac{1}{2} [\tan(x + h) + \tan(x - h)] - \tan x = \frac{\tan x \cdot \sin^2 h}{\cos^2 x - \sin^2 h}$

ix) $\tan x = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$

x) $\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A + 1} = \tan 2A$

xi) $\sin^4 \theta + \cos^4 \theta = \frac{1}{4} (\cos 4\theta + 3)$

xii) $4 \sin^3 A \cdot \cos 3A + 4 \cos^3 A \cdot \sin 3A = 3 \sin 4A$

xiii) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

xiv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

xv) $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

xvi) $\cosh(u + v) = \cosh u \cdot \cosh v + \sinh u \cdot \sinh v$

and then verify

$$\cosh(u - v) = \cosh u \cdot \cosh v - \sinh u \cdot \sinh v$$

xvii) $\cosh u \cdot \sinh v = \frac{1}{2} [\sinh(u + v) - \sinh(u - v)]$

xviii) $\sinh u \cdot \sinh v = \frac{1}{2} [\cosh(u + v) - \cosh(u - v)]$

xix) $\cosh 3u = \cosh u + 4 \sinh^2 u \cdot \cosh u = 4 \cosh^3 u - 3 \cosh u$

xx) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

10. If $u = \frac{1 + \sin \theta}{\cos \theta}$, prove that $\frac{1}{u} = \frac{1 - \sin \theta}{\cos \theta}$ and deduce formula for $\sin \theta$, $\cos \theta$, $\tan \theta$ in terms of u . (ans.: $(u^2 - 1)/(u^2 + 1)$; $2u/(u^2 + 1)$; $(u^2 - 1)/(u^2 + 1)$)

11. If $\sin(x + \alpha) = 2\cos(x - \alpha)$; prove that : $\tan x = \frac{2 - \tan \alpha}{1 - 2 \tan \alpha}$.

12. If $\sin(x - \alpha) = \cos(x + \alpha)$; prove that : $\tan x = 1$.

13. If $x = \cos \theta + \cos 2\theta$ and $y = \sin \theta + \sin 2\theta$. Show that :

i) $x^2 - y^2 = \cos 2\theta + 2\cos 3\theta + \cos 4\theta$

ii) $2xy = \sin 2\theta + 2\sin 3\theta + \sin 4\theta$

14. If $\cos 2A \cdot \cos 2B = \cos 2\theta$, prove that :

$\sin^2 A \cdot \cos^2 B + \cos^2 A \cdot \sin^2 B = \sin^2 \theta$

15. If $S = \sin \theta$ and $C = \cos \theta$, simplify :

i) $\frac{S \cdot C}{\sqrt{1 - S^2}}$, ii) $\frac{S \cdot \sqrt{1 - S^2}}{C \cdot \sqrt{1 - C^2}}$, iii) $\frac{C}{S} + \frac{S}{C}$

(ans.:i) $\sin \theta$; ii) 1; iii) $\sec \theta \cdot \csc \theta$)

16. Eliminate θ from the following equations :

i) $x = a \cdot \csc \theta$ and $y = b \cdot \sec \theta$

ii) $x = \sin \theta + \cos \theta$ and $y = \sin \theta - \cos \theta$

iii) $x = \sin \theta + \tan \theta$ and $y = \sin \theta - \tan \theta$

iv) $x = \tan \theta$ and $y = \tan 2\theta$

(ans.:i) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$; ii) $x^2 + y^2 = 2$; iii) $\frac{4}{(x+y)^2} - \frac{4}{(x-y)^2} = 1$; iv) $y = \frac{2x}{1-x^2}$)

17. In the acute – angled triangle OPQ , the altitude OR makes angles A and B with OP and OQ . Show by means of areas that if $OP=q$, $OQ=p$, $OR=r$: $p \cdot q \cdot \sin(A+B) = q \cdot r \cdot \sin A + p \cdot r \cdot \sin B$.

18. Given that $\alpha = \sin^{-1} \frac{1}{2}$, find $\cos \alpha$, $\tan \alpha$, $\sec \alpha$, and $\csc \alpha$.

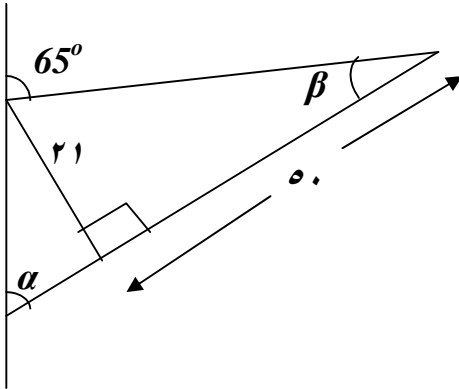
(ans.: $\frac{\sqrt{3}}{2}$; $\frac{1}{\sqrt{3}}$; $\frac{2}{\sqrt{3}}$; 2)

19. Evaluate the following expressions :

a) $\text{Sin}(\text{Cos}^{-1} \frac{1}{\sqrt{2}})$ b) $\text{Csc}(\text{Sec}^{-1} 2)$
 c) $\text{Cot}(\text{Cos}^{-1} 0)$ d) $\text{Sin}^{-1} 1 - \text{Sin}^{-1}(-1)$
 e) $\text{Cos}(\text{Sin}^{-1} 0.8)$ f) $\text{Cos}^{-1}(-\text{Sin} \frac{\pi}{6})$

(ans.: $1/\sqrt{2}; 2/\sqrt{3}; 0; \pi; 0.6; 2\pi/3$)

20. Find the angle α in the below graph (Hint : $\alpha + \beta = 65^\circ$) :



(ans.: 42.2)

21. Let $\text{Sech } u = 3/5$, determine the values of the remaining five hyperbolic functions .

(ans.: $\text{Cosh } u = 5/3; \text{Sinhu} = \mp 4/3; \text{tanh } u = \mp 4/5; \text{Cothu} = \mp 5/4; \text{Cschu} = \mp 3/4$)

22. Rewrite the following expressions in terms of exponentials , write the final result as simply as you can :

a) $\text{Sinh}(2 \ln x)$ b) $\frac{1}{\text{Cosh } x - \text{Sinh } x}$
 c) $\text{Cosh } 3x - \text{Sinh } 3x$ d) $\ln(\text{Cosh } x + \text{Sinh } x) + \ln(\text{Cosh } x - \text{Sinh } x)$
 (ans.: $(x^4 - 1)/(2x^2); e^x; e^{-3x}; 0$)

23. Solve the equation for x ; $\text{tanh } x = 3/5$. (ans.: $\ln 2$)

24. Show that the distance r from the origin O to the point $P(\text{Cosh } u, \text{Sinhu})$ on the hyperbola $x^2 - y^2 = 1$ is $r = \sqrt{\text{Cosh } 2u}$.

25. If θ lies in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\text{Sinh } x = \tan \theta$. Show that :

$\text{Cosh } x = \text{Sec } \theta$, $\text{tanh } x = \text{Sin } \theta$, $\text{Coth } x = \text{Csc } \theta$, $\text{Csch } x = \text{Cot } \theta$, and $\text{Sech } x = \text{Cos } \theta$.

26. Derive the formula : $\text{tanh}^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$; $|x| < 1$

27. Find : $\lim_{x \rightarrow \infty} [\text{Cosh}^{-1} x - \ln x]$. (ans.: $\ln 2$)