Higher Order Derivative

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$$

$$\frac{\partial^n z}{\partial x^n} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\dots z\right)\right)\dots}_{n\text{-times}}$$

mixed derivative

$$\frac{\partial^2 z}{\partial x \partial y} = \underleftarrow{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} z\right)}_{R.\ to\ L} = \underrightarrow{z_{yx}}_{L\ to\ R}$$

<u>note</u> if f and it partial derivative are continuous function. Then

$$f_{xy} = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

<u>ex</u> find z_{xyyx} when $z = x^3y^2 + xy + y^2$

$$z_{xyyx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(x^3y^2 + xy + y^2\right)\right)\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(2x^2y^2 + y\right)\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(4x^2y + 1\right)\right)$$

$$= \frac{\partial}{\partial x}\left(4x^2\right) = 8x$$

ex $w = e^{y}\cos(x)$

find (a) w_{xyy} (b) w_{yxx}

a= $(e^{y}\cos x)_{xyy}$

$= (-e^{y}\sin x)_{yy}$

$= (-e^{y}\sin x)_{y}$

$= -e^{y}\sin x$

b $\equiv (e^{y}\cos x)_{yxx}$

$= (e^{y}\cos x)_{xx}$

$= (-e^{y}\sin x)_{x}$

$= -e^{y}\cos x$

ex $f(x,y) = y^{3}e^{-5x}$ find f_{yyxx} at $(0,1)$

sol.

$f_y = 3y^{2}e^{-5x}$

$f_{yy} = 6y\,e^{-5x}$

$f_{yyx} = -30\,y\,e^{-5x}$

$f_{yyxx} = 150\,y\,e^{-5x}$

$f_{yyxx}(0,1) = 150\,(1)\,.\,e^{0} = 150$

$\square$

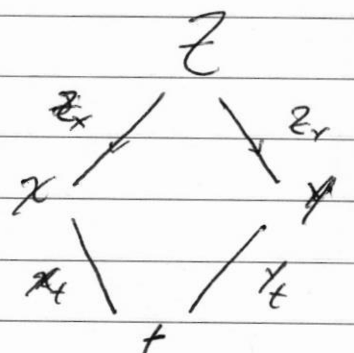
Chain Rule

Chain Rule for partial derivative have ~~two~~ Three form

(A)

$z = f(x, y)$ and $x = x(t)$ & $y = y(t)$

find $z_t = \frac{\partial z}{\partial t}$

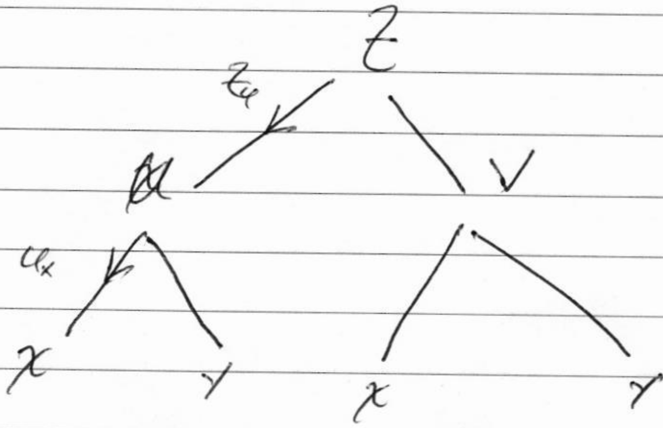


$$z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

(B) if $z = f(u, v)$ and $u = h(x, y)$ $v = g(x, y)$

$$\frac{\partial z}{\partial x} = z_u u_x + z_v v_x$$

$$\frac{\partial z}{\partial y} = z_u u_y + z_v v_y$$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

(c) $z = f(u) \quad , \quad u = g(x, y)$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$\begin{tikzcd} & z \arrow[d] & \\ & u \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

ex for (c)

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ & show That $x z_y + y z_x = 0$

sol. when $z = f(x^2 - y^2)$

let $u = x^2 - y^2 \Rightarrow z = f(u)$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot (2x) = z' \, 2x$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} (-2y) = -z' \, 2y$$

$$\Rightarrow 2xy z' + (-2xy z') = 0$$

H.W. $R = \sqrt{x^2 + y^2}$ ~~show That~~ Find $\frac{\partial R}{\partial x}$, $\frac{\partial^2 R}{\partial y^2}$

$\frac{x}{R}$ " " $\frac{x^2}{R^3}$

ex (B)

Let $z = e^{uv}$, $u = 2x + y$, $v = \frac{x}{y}$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

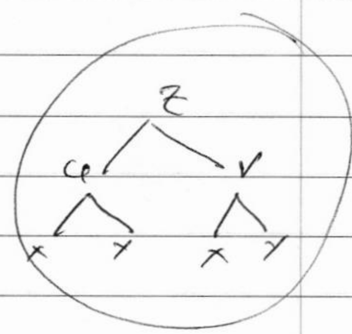
sol.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= v e^{uv} \cdot 2 + u e^{uv} \cdot \frac{1}{y}$$

$\frac{\partial z}{\partial y}$ = H.w. class work

ex2 : $z = f([x-y], [-x+y])$



Show That $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ $(z_x + z_y = 0)$

sol. let $u = x - y$ & $v = x + y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= z_u \cdot 1 + z_v \cdot (-1) = z_u - z_v \quad \text{—— } ①$$

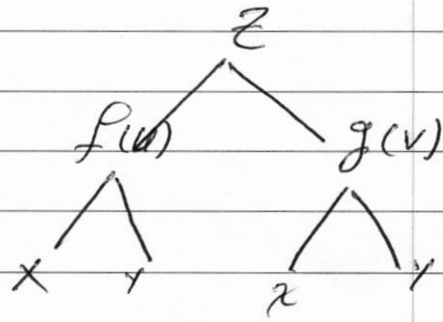
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= z_u \cdot (-1) + z_v \cdot (1) = -z_u + z_v \quad \text{—— } ②$$

$\therefore z_x + z_y = 0$

ex $\quad z = f(y+cx) + g(y-cx) \qquad c \neq 0$

Show That
$$\frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial^2 z}{\partial y^2}$$



Sol.

Let $u = y + cx$

$\quad\;\; v = y - cx$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= f'(u) \cdot (c) + g'(v) \cdot (-c)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = c \frac{\partial}{\partial x}\left(f'(u) - g'(v)\right)$$

$$= c\left[\frac{\partial f'(u)}{\partial u} \cdot \frac{\partial u}{\partial x} - \frac{\partial g'(v)}{\partial v} \cdot \frac{\partial v}{\partial x}\right]$$

$$= c\left[f''(u) \cdot c - g''(v) \cdot (-c)\right]$$

$$= c^2\left[f''(u) + g''(v)\right] \quad \text{———} \; ①$$

$$\frac{\partial z}{\partial y} = f'(u) \cdot 1 + g'(v) \cdot 1 = f' + g'$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[f'(u) + g'(v)\right] = f'' \cdot 1 + g'' \cdot 1$$

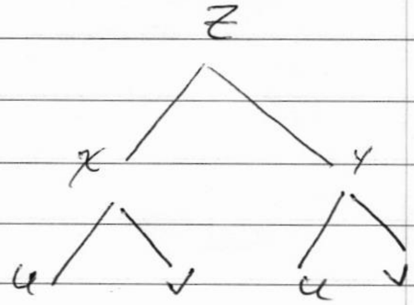
$$= f''(u) + g''(v)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial^2 z}{\partial y^2} \qquad \text{Q.E.D.} \; \square$$

ex $\quad z = \sin xy, \quad x = u + v, \quad y = u \cdot v$

Show That :-

$$\frac{\partial^2 z}{\partial u \partial x} = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y \partial x} \cdot \frac{\partial y}{\partial u}$$



sol. L.H.S $\quad \frac{\partial z}{\partial x} = y \cos xy$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(y \cos xy\right)$$

$$\circledast \quad = \frac{\partial (y \cos xy)}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial (y \cos xy)}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= -y^2 \sin xy \cdot (1) + [\cos xy - yx \sin xy]\, v \quad ①$$

R.H.S.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(y \cos xy) = -y^2 \sin xy$$

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(y \cos xy) = [\cos xy - yx \sin xy]$$

$$\frac{\partial y}{\partial u} = v$$

$$\Rightarrow ① = ② \Rightarrow \text{Q.E.D.} \ \square$$

note from $\circledast$ one can see if we replace ($y \cos xy$ by $\frac{\partial z}{\partial x}$) we get the sol. without long detailes ....! so let $\begin{cases} z = f(x, y) \\ x = g(u, v) \\ y = h(u, v) \end{cases}$

resol. above ex ....

Total Differential (extend of Chain Rule)

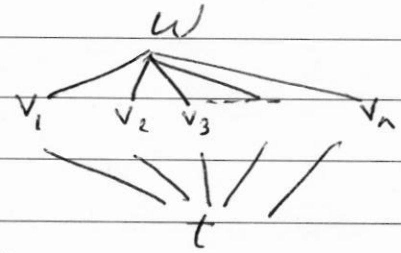
if $w = f(V_1, V_2, \ldots, V_n)$ Then

$w' =$

$$\frac{\partial w}{\partial V_1} \cdot dV_1 + \frac{\partial w}{\partial V_2} dV_2 + \cdots + \frac{\partial w}{\partial V_n} \cdot dV_n$$

and if $V_1, \ldots, V_n$ is a functions of t Then

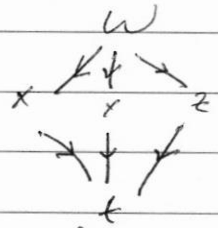
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial V_1} \cdot \frac{dV_1}{dt} + \frac{\partial w}{\partial V_2} \cdot \frac{dV_2}{dt} + \cdots + \frac{\partial w}{\partial V_n} \cdot \frac{dV_n}{dt}$$



ex! Let $w = xy + yz$

$x = t^2$, $y = \sin t$, $z = e^t$,

find $\frac{\partial w}{\partial t}$?



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= y \cdot 2t + (x+z)\cos t + y \cdot e^t$$

$$= 2t \sin t + (t^2 + e^t)\cos t + e^t \sin t$$

note

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial V_1}, \frac{\partial w}{\partial V_2}, \ldots, \frac{\partial w}{\partial V_n} \right\rangle \cdot \left\langle \frac{dV_1}{dt}, \ldots, \frac{dV_n}{dt} \right\rangle$$

and

$$\frac{\partial w}{\partial r} = \left\langle \frac{\partial w}{\partial V_1}, \ldots \frac{\partial w}{\partial V_n} \right\rangle \cdot \left\langle \frac{\partial V_1}{\partial r}, \ldots, \frac{\partial V_n}{\partial r} \right\rangle$$

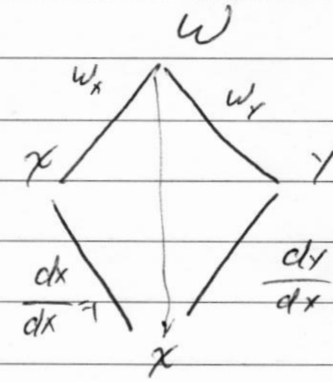
$$\frac{\partial w}{\partial s} = \left\langle \frac{\partial w}{\partial V_1}, \ldots \frac{\partial w}{\partial V_n} \right\rangle \cdot \left\langle \frac{\partial V_1}{\partial s}, \ldots, \frac{\partial V_n}{\partial s} \right\rangle$$

Implicit Differentiation (partial)

Case 1

$w = F(x,y) = 0$

and $y = h(x)$



$$\frac{dw}{dx} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dx}$$

$$0 = \frac{\partial w}{\partial x}\cdot 1 + \frac{\partial w}{\partial y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial w/\partial x}{\partial w/\partial y} = \frac{-F_x}{F_y}$$

ex1 find $\frac{dy}{dx}$ if $x^2 + \sin y - 2y = 0$

sol.

$$w = F(x,y) = x^2 + \sin y - 2y = 0$$

$$\Longrightarrow \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2x}{\cos y - 2}$$

Case 2 when $w = F(x,y) \neq 0$ or in General

$$w = F(x,y,z,\dots) \neq 0$$

$$w = F(v_1, v_2, \dots, v_n) \neq 0$$

Then we can find $\frac{dw}{dv_j}$ when all of $v_1, v_2, v_3, \dots$

can be write as a function of v_j

For ex ~~in R^3~~

$$\frac{dw}{dx} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dx}$$

$$\frac{dw}{dy} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dy} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dy} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dy}$$

same way

ex 2

if $w = \sqrt{x^2 + y^2 + z^2}$ $\quad x = \cos 2y$, $z = \sqrt{y}$

find ⓐ $\frac{dw}{dy}$, $\frac{dw}{dx}$, $\frac{dw}{dz}$

sol.

ⓐ

$$\frac{dw}{dy} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dy} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dy}$$

$$= -2x\, w^{-1} \cdot \sin 2y + y w^{-1} \cdot 1 + \cancel{z} w^{-1} \cdot \frac{1}{2} \cancel{y^{-1}}$$

$$= w^{-1} \left[-2x \sin 2y + y + \frac{1}{2} \right]$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \left[-2x \sin 2y + y + \frac{1}{2} \right]$$

ⓑ H.w.

ⓒ $y = z^2 \Rightarrow x = \cos 2z^2$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dz} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dz} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dz}$$

$$= -4zx\, w^{-1} \sin 2z^2 + 2zy\, w^{-1} + z w^{-1} \cdot 1$$

$$= z w^{-1} \left(-4x \sin 2z^2 + 2y + 1 \right)$$

$$= \frac{z}{\sqrt{x^2+y^2+z^2}} \left(-4x \sin 2z^2 + 2y + 1 \right)$$