# Gradient vectors

2016 - 11 - 2

and

# Directional Derivatives (D.D.)

Def.

If the partial derivative of the function $f(x,y,z)$ define at $P_0 = (x_0, y_0, z_0)$, Then the "Gradient" of $f$ is the vector

$$\nabla f = f_x i + f_y j + f_z K = \langle f_x , f_y , f_z \rangle$$

at $P_0 = (x_0, y_0, z_0)$

$$(\nabla f)_{P_0} = (\langle f_x , f_y , f_z \rangle)_{P_0 (x_0, y_0, z_0)}$$

and if $f$ has cont' partial derivative at $P_0$ ; and if $\vec{u}$ is the unit vector Then the Derivative of $f$ in the Direction of $u$ is :-

$$D_u f = \overrightarrow{\nabla f} \cdot \vec{u}$$

$$(D_u f)_{P_0} = (\overrightarrow{\nabla f})_{P_0} \cdot \vec{u}$$

note

$\nabla f$ read (Gredient or Grad or Del) of $f$

**ex1**

Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ in the direction of $\vec{A}$, at $P_0 = (1,1,0)$, where $\vec{A} = \langle 2,-3,6 \rangle$

**sol.**

$$\nabla f = \langle 3x^2 - y^2 \,,\, -2xy \,,\, -1 \rangle$$

$$\vec{u} = \frac{A}{\|A\|} = \frac{\langle 2,-3,6\rangle}{7} = \left\langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \right\rangle$$

$$(D_u f)_{P_0} = \nabla f_{P_0} \cdot \vec{u}$$

$$= \langle 2,-2,-1 \rangle \cdot \left\langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \right\rangle$$

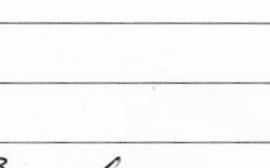
$$= \frac{4+6-6}{7} = \frac{4}{7}$$

**ex2**

Find D.D. of $f(x,y) = e^{xy}$ at $P = (-2,0)$ in the direction of $\vec{A}$ that make an angle of $\frac{\pi}{3}$ with with +ve $x$ axis?

**sol.**

$$D_u f = f_x \cos\theta + f_y \sin\theta$$




$$D_u f\big|_{(-2,0)} = y e^{xy} \cos\frac{\pi}{3} + x e^{xy} \sin\frac{\pi}{3}\Big|_{(-2,0)}$$

$\cos\theta = \frac{u_1}{1}$

$\sin\theta = \frac{u_2}{1}$

$$= 0 \cdot \_\_\_ + (-2)e^{0} \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

# Increment and Distance

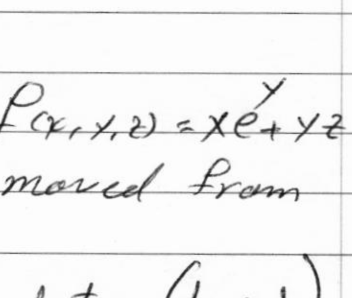
To Estaimate the change of $f$ in the diraction of $\vec{u}$; i.e. The change of The value of $f$ when we move a small distance "ds" from the point $P_0$ in particalal direction $\vec{u}$, use The formula:-

$$df = \left(\nabla f \cdot \vec{u}\right)_{P_0} \cdot ds$$

ex

Estimate how much The value of $f(x,y,z) = xe^{y} + yz$ will change if The point $P(x,y,z)$ is moved from $P_0(2,0,0)$ toward $P_1(4,1,-2)$ a distance $(ds = 0.1)$

sol. $u = \dfrac{\overrightarrow{P_0P_1}}{\|P_0P_1\|} = \dfrac{\langle 2, 1, -2\rangle}{3} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right\rangle$




$\nabla f = \langle e^{y}, xe^{y}, z\rangle$

$\nabla f \cdot u|_{P_0} = \langle 1, 2, 0\rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right\rangle = \frac{2+2+0}{3} = \frac{4}{3}$

$df = \nabla f \cdot u|_{P_0} \cdot ds = \frac{4}{3} * 0.1 = 0.13$

notes

$D.D = D_u f = \nabla f \cdot u$

$= |\nabla f| \cdot |u| \cdot \cos\theta$

$= |\nabla f| \cdot \cos\theta$

so

1) D.D. has its Largest Value when $\cos\theta = 1$
2) " " " smallest " " $\cos\theta = 0$

3) $f$ increses most rapidlly at any point in the direction of $\nabla f$. ( $D.D = |\nabla f| \cdot \cos 0 = |\nabla f|$

and decreses most rapidlly at any point in the direction of $-\nabla f$ ( $D.D = |\nabla f| \cdot \cos\pi = -|\nabla f|$

4) $D_{-u} f = \nabla f \cdot (-u) = -D_u f$

ex

Given That $DD = 5$ for $f(x,y,z)$ at $(3,-2,1)$ in dirction of $\langle 2,-1,-2 \rangle$; find Grident of $f$ ?

soln

$D.D = \nabla f \cdot \vec{u}$

$A = \langle 2,-1,-2 \rangle$

$\vec{u} = \frac{\langle 2,-1,-2 \rangle}{\sqrt{9}} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle$

$DD \cdot \vec{u} = \nabla f \cdot \vec{u} \cdot \vec{u}$

$5 \cdot \vec{u} = \nabla f$

$\langle \frac{10}{3}, \frac{-5}{3}, \frac{-10}{3} \rangle = \nabla f_{p}$

# Maximum, Minimum

## Saddle point.

def.

Let $R_g$ be the ragin bounded by

$$R_g = \{ (x,y) \in \mathbb{R}\times\mathbb{R} \mid x_0 \le x \le x_1 ,\ y_0 \le y \le y_1 \}$$

The function $f: \overset{R\times R}{\cancel{R_g}} \to R$ is said to be have

Local maxima at point $(x_0,y_0)$ if $f(x_0,y_0) \geq f(x,y)\ \forall x,y \in R_g$

&

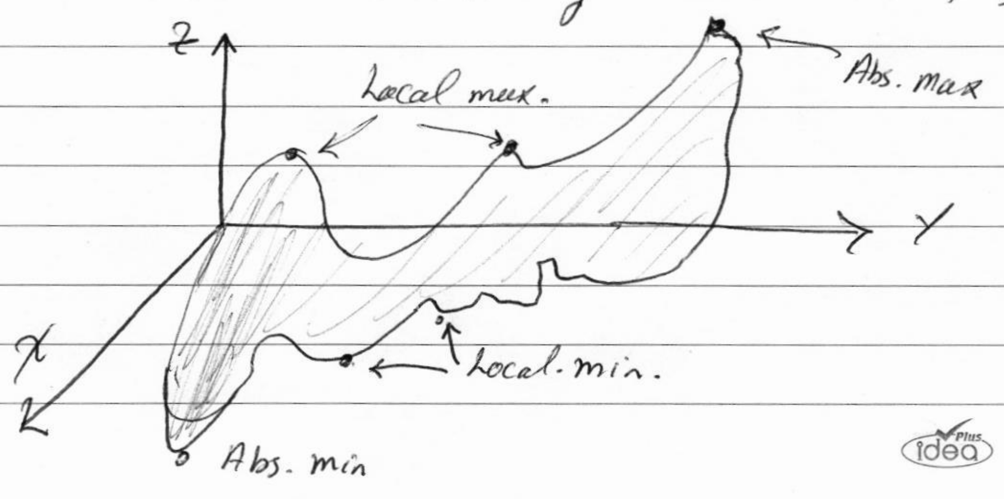
$$f(x_0,y_0) \geq f(x,y) \quad \forall x,y \in R_g$$

Local minma at point $(x_0,y_0)$ if $f(x_0,y_0) \leq f(x,y)\ \forall x,y \in R_g$

$$f(x_0,y_0) \leq f(x,y) \quad \forall x,y \in R_g$$

The ward Local will be replaced by

Global or Absolut when $R_g$ = Domain of $f$



# The 2nd partial Derivative Test

Let $f$ be a function of two variable with, 2nd cont's order partial derivative in some Rg ; and let

$$D = f_{xx} f_{yy} - f_{xy}^2 \Big|_{(x_0, y_0)} , \quad E = f_{xx} \Big|_{(x_0, y_0)}$$

Then The critical point $(x_0, y_0)$ is :-

1) $(x_0, y_0)$ Local maxima if $D > 0$ & $E < 0$

2) $(x_0, y_0)$ Local minima if $D > 0$ & $E > 0$

3) $(x_0, y_0)$ Saddle point if $D < 0$

4) $(x_0, y_0)$ No information if $D = 0$

The critical pt. can be found from the first derivative by equalizer to zero.

ex

Locate The Local extrems and Saddle pt for

$$f(x, y) = 4xy - x^4 - y^4$$

~~2,6 ++ min~~

sol.

$$f_x = 4y - 4x^3 \Rightarrow y = x^3$$
$$f_y = 4x - 4y^3 \Rightarrow x = y^3$$

$$\Rightarrow (0,0),\ (1,1),\ (-1,-1)$$

are the critical pts

now

$f_{xx} = -12x^2$

$f_{yy} = -12y^2$

$f_{xy} = 4$

$\Rightarrow$

| pt | E ($f_{xx}\|_{(x,y)}$) | $f_{xx}f_{yy} - f_{xy}^2 = D$ | result |
|---|---|---|---|
| (0,0) | 0 | $0\times0-16 = -16$ | saddle |
| (1,1) | $-12$ | $144-16 = 128$ | maxima |
| (-1,-1) | $-12$ | $144-16 = 128$ | maxima |

ex (H.W, 2,6 ++ min) Local Extremes for $f(x,y) =$

$$3x^2 - 2xy + y^2 - 8y$$